Multiplication and Integral Operators on Banach Spaces of Analytic Functions

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Chapter 1

Introduction

We investigate operators on Banach spaces of analytic functions on the unit disk D in the complex plane. The operator T_g, with symbol $g(z)$ an analytic function on the disk, is defined by

$$T_g f(z) = \int_0^z f(w)g'(w)\,dw \qquad (z \in D).$$

T_g is a generalization of the standard integral operator, which is T_g when $g(z) = z$. Letting $g(z) = \log(1/(1-z))$ gives the Cesáro operator [1]. Discussion of the operator T_g first arose in connection with semigroups of composition operators (see [16] for background). Characterizing the boundedness and compactness of T_g on certain spaces of analytic functions is of recent interest, as seen in [2], [3], [7] and [16], and open problems remain. T_g and its companion operator $S_g f(z) = \int_0^z f'(w)g(w)\,dw$ are related to the multiplication operator $M_g f(z) = g(z)f(z)$, since integration by parts gives

$$M_g f = f(0)g(0) + T_g f + S_g f. \tag{1.1}$$

If any two of M_g, S_g, and T_g are bounded, then so is the third. But on many spaces, there exist functions g for which one operator is bounded and two are unbounded. The pointwise multipliers of the Hardy, Bergman and Bloch spaces are well known, as well as David Stegenga's results on multipliers of the Dirichlet space and $BMOA$. Theorem 2.7 below states these results. We examine boundedness and compactness of T_g and S_g on the all these spaces. According to [2], boundedness of the operator T_g on H^2 was first characterized by Christian Pommerenke. Boundedness and compactness of T_g was characterized on the Hardy spaces H^p for $p < \infty$ by Alexandru Aleman and Joseph Cima in [2], and on the Bergman spaces by Aleman and Aristomenis Siskakis in [3]. In [16], Siskakis and Ruhan Zhao proved T_g is bounded (and compact) on $BMOA$ if and only if $g \in LMOA$. As seen in sections 3.1, 3.2, and 3.3, boundedness of S_g is equivalent to g being bounded, while the conditions for T_g are more complicated.

An interesting interplay of the three operators M_g, T_g, and S_g occurs. In characterizing the multipliers of the Dirichlet and Bloch spaces and $BMOA$, two conditions on g are required. It turns out that the operators T_g and S_g split the conditions on the multipliers. One condition characterizes boundedness of T_g, and the other condition characterizes when S_g is bounded. In the case of the Hardy and Bergman spaces, the condition for T_g to be bounded subsumes that for S_g and M_g. Action on the space H^∞ provides an example in which M_g is bounded while T_g and S_g are not. This phenomenon is unique among the other spaces studied here, and a complete characterization of the symbols that make T_g and S_g bounded on H^∞ is unknown.

We also examine conditions on the symbol g that cause T_g and S_g to have closed range on certain spaces. We examine aspects of the problems on Hardy, weighted Bergman, and Bloch spaces, and $BMOA$. On the spaces studied, T_g and S_g have closed range if and only if they are bounded below (Theorem 2.9). In Theorem 4.7, we characterize the symbols g for which S_g is bounded below on the Bloch space. We also point out analogous results for the Hardy space H^2 and the weighted Bergman spaces A^p_α for $1 \leq p < \infty, \alpha > -1$. In Theorem 4.1 we show the companion operator T_g is never bounded below on H^2, Bloch, nor $BMOA$. We subsequently mention an example from [15] demonstrating T_g may be bounded below on A^p.

Chapter 2

Background

2.1 General Preliminaries

For two nonnegative quantities f and g, the notation $f \lesssim g$ will mean there exists a universal constant C such that $f \leq Cg$. $f \sim g$ will mean $f \lesssim g \lesssim f$.

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane and $H(D)$ the set of analytic functions on D.

Theorem 2.2 is a generalization of a result on multipliers of Banach spaces in which point evaluation is a bounded linear functional. We state the result for multipliers first.

Theorem 2.1. *Let X be a Banach space of analytic functions on which point evaluation is bounded for each point $z \in D$. Suppose M_g is bounded on X for some $g \in X$. Then*

$$|g(z)| \leq \|M_g\|.$$

The proof is similar to Theorem 2.2, so we omit it here. (See, e.g., [5, Lemma

11].)

Theorem 2.2. *Let X and Y be Banach spaces of analytic functions, $z \in D$, and let λ_z^0 and λ_z^1 be linear functionals defined by $\lambda_z^0 f = f(z)$ and $\lambda_z^1 f = f'(z)$ for $f \in X \cup Y$. Suppose λ_z^0 and λ_z^1 are bounded.*

(i) *If S_g maps X boundedly into Y, then*

$$|g(z)| \leq \|S_g\| \frac{\|\lambda_z^1\|_Y}{\|\lambda_z^1\|_X}.$$

(ii) *If T_g maps X boundedly into Y, then*

$$|g'(z)| \leq \|T_g\| \frac{\|\lambda_z^1\|_Y}{\|\lambda_z^0\|_X}.$$

Proof. Note that, for $f \in X$,

$$|f'(z)||g(z)| = |\lambda_z^1 S_g(f)| \leq \|\lambda_z^1\|_Y \|S_g\| \|f\|_X.$$

Since

$$\sup_{\|f\|_X=1} |f'(z)| = \|\lambda_z^1\|_X,$$

taking the supremum of both sides over all f in X with norm 1 gives us

$$\|\lambda_z^1\|_X |g(z)| \leq \|S_g\| \|\lambda_z^1\|_Y.$$

Hence (i) holds. Similarly,

$$|f(z)||g'(z)| = |\lambda_z^1 T_g(f)| \leq \|\lambda_z^1\|_Y \|T_g\| \|f\|_X.$$

Taking the supremum over $\{f \in X : \|f\|_X = 1\}$, we get

$$\|\lambda_z^0\|_X |g'(z)| \le \|T_g\| \|\lambda_z^1\|_Y.$$

This completes the proof. $\square$

When $Y = X$, we obtain the following corollary.

Corollary 2.3. *If X is a Banach space of analytic functions on which point evaluation of the derivative is a bounded linear functional, and S_g is bounded on X, then g is bounded.*

Corollary 2.3 will be used frequently below, because λ_z^1 is bounded for each $z \in D$ on the spaces in which we are interested.

2.2 Spaces of Analytic Functions

For $1 \le p < \infty$, the Hardy space H^p on D is

$$\{f \in H(D) : \|f\|_p = \sup_{0<r<1} \int_0^{2\pi} |f(re^{it})|^p \, dt < \infty\}.$$

The space of bounded analytic functions on D is

$$H^\infty = \{f \in H(D) : \|f\|_\infty = \sup_{z \in D} |f(z)| < \infty\}.$$

We define weighted Bergman spaces, for $\alpha > -1$, $1 \le p < \infty$,

$$A^p_\alpha = \{f \in H(D) : \|f\|_{A^p_\alpha} = \int_D |f(z)|^p (1 - |z|^2)^\alpha \, dA(z) < \infty\},$$

where $dA(z)$ refers to Lebesgue area measure on D. We denote the unweighted Bergman space $A^p = A^p_0$.

The Bloch space is

$$\mathcal{B} = \{f \in H(D) : \|f\|_{\mathcal{B}} = \sup_{z\in D} |f'(z)|(1-|z|^2) < \infty\}.$$

Note that $\| \|_{\mathcal{B}}$ is a semi-norm. The true norm accounts for functions differing by an additive constant. It is well known that H^∞ a subspace of $\mathcal{B}$, and $\|f\|_{\mathcal{B}} \le \|f\|_\infty$ for all $f \in H^\infty$ [20, Proposition 5.1]. For $\alpha > 0$, the α-Bloch space $\mathcal{B}_\alpha$ and logarithmic Bloch space $\mathcal{B}_{\alpha,\ell}$ are the following sets of analytic functions defined on the disk.

$$\mathcal{B}_\alpha = \{f \in H(D) :\ \|f\|_{\mathcal{B}_\alpha} = \sup_{z\in D} |f'(z)|(1-|z|^2)^\alpha < \infty\}, \text{ and}$$

$$\mathcal{B}_{\alpha,\ell} = \{f \in H(D) :\ \|f\|_{\mathcal{B}_{\alpha,\ell}} = \sup_{z\in D} |f'(z)|(1-|z|^2)^\alpha \log\frac{1}{1-|z|} < \infty\}.$$

Since $\mathcal{B} = \mathcal{B}_1$, we define $\mathcal{B}_\ell := \mathcal{B}_{1,\ell}$. For $0 < \alpha < 1$, $\mathcal{B}_\alpha$ are (analytic) Lipschitz class spaces (see [8, Theorem 5.1]).

Define the conelike region with aperture $\alpha \in (0,1)$ at $e^{i\theta}$ to be

$$\Gamma_\alpha(e^{i\theta}) = \left\{z \in D : \frac{|e^{i\theta} - z|}{1-|z|} < \alpha\right\}.$$

For a function f on D, define the *nontangential limit* of f at $e^{i\theta}$ to be

$$f^*(e^{i\theta}) = \lim_{z\to e^{i\theta}} f(z) \qquad (z \in \Gamma_\alpha(e^{i\theta})),$$

provided the limit exists. If $f \in H^1$, then the nontangential limit of f exists for almost all $e^{i\theta} \in \partial D$ (see [10, Ch. I, Theorem 5.2]). In this case we associate f with

f^*, so that we have a function f defined on $\overline{D}$ except for a set of measure 0 in ∂D.

For a measurable complex-valued function φ defined on ∂D, and an arc $I \subseteq \partial D$, define

$$\varphi_I = \frac{1}{|I|} \int_I \varphi(e^{it})\, dt,$$

where $|I|$ is the length of I, normalized so that $|I| \leq 1$. The function φ has *bounded mean oscillation* if

$$\sup_I \frac{1}{|I|} \int_I |\varphi(e^{it}) - \varphi_I| dt < \infty,$$

as I ranges over all arcs in ∂D.

The space $BMOA$ is the set analytic functions f on D such that f^* has bounded mean oscillation. Associating f with f^*, we define the semi-norm

$$\|f\|_* = \sup_I \frac{1}{|I|} \int_I |f(e^{it}) - f_I| dt.$$

Of course, f^* must be defined almost everywhere, so assume $f \in H^1$, although in fact, $BMOA \subset H^2$. One way to see this is via the duality relations $(H^1)^* \cong BMOA$ (see [10, Exercise 5, p. 261]) and $(H^p)^* \cong H^q$, where $1/p + 1/q = 1$ (see [20, 8.1.8]). Since $H^2 \subset H^1$, we have $BMOA \cong (H^1)^* \subset (H^2)^* \cong H^2$. In fact, for all $f \in BMOA$,

$$\|f\|_2 \lesssim \|f\|_*. \tag{2.1}$$

Note that H^∞ is a subspace of $BMOA$, since the following calculation shows

$$\|f\|_* \leq \|f\|_\infty. \tag{2.2}$$

$$
\begin{aligned}
\|f\|_* &= \sup_I \frac{1}{|I|}\int_I |f - f_I| \\
&\le \sup_I \left(\frac{1}{|I|}\int_I |f - f_I|^2\right)^{1/2} \\
&= \sup_I \left(\frac{1}{|I|}\int_I (f - f_I)(\overline{f} - \overline{f_I})\right)^{1/2} \\
&= \sup_I \left(\frac{1}{|I|}\left(\int_I |f|^2 - \int_I f\overline{f_I} - \int_I \overline{f} f_I\right) + |f_I|^2\right) \\
&= \sup_I (1/|I|)((|f|^2)_I - |f_I|^2)^{1/2} \\
&\le \sup_I (1/|I|)|f|_I \le \|f\|_\infty.
\end{aligned}
$$

The function $f \in BMOA$ has *vanishing mean oscillation* if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|I| < \delta$ implies

$$\frac{1}{|I|}\int_I |f - f_I| < \varepsilon.$$

The subspace of $BMOA$ consisting of the functions with vanishing mean oscillation is denoted $VMOA$. Another way we write the condition defining $VMOA$ is

$$VMOA = \{f \in BMOA : \lim_{|I|\to 0} \frac{1}{|I|}\int_I |f - f_I| = 0\}.$$

A noteworthy subspace of $VMOA$ is those functions whose mean oscillation vanishes more quickly than $1/\log(1/|I|)$. We define

$$LMOA = \{f \in VMOA : \lim_{|I|\to 0} \frac{\log(1/|I|)}{|I|}\int_I |f - f_I| = 0\}.$$

A useful characterization of $BMOA$ for our purposes involves Carleson measures.

For $1 \le p < \infty$, a complex measure μ on D is a *Carleson measure for H^p* if

$$\int_D |f|^p \, d\mu \lesssim \|f\|_p^p.$$

For an arc $I \subseteq \partial D$, define the *Carleson rectangle* associated with I to be

$$S(I) = \{re^{i\theta} : 1 - |I| < r < 1, e^{i\theta} \in I\}.$$

The measure μ is Carleson for H^p if and only if there exists $C > 0$ such that $\mu(S(I)) \le C|I|$ for all arcs $I \subseteq \partial D$ (a well known result of Lennart Carleson, see [10, Theorem II.3.9]). The smallest such C is called the Carleson constant for the measure μ. Define, for $f \in H(D)$, $d\mu_f(z) = |f'(z)|^2(1 - |z|^2) \, dA(z)$. $BMOA$ is the set of f for which μ_f is Carleson for H^p, and the $BMOA$ semi-norm $\|f\|_*$ is comparable to the square root of the Carleson constant for μ_f (see [10, Ch. VI, Sec. 3]. The space $VMOA$ is the set of f for which

$$\lim_{|I| \to 0} \frac{\mu_f(S(I))}{|I|} = 0.$$

Also,

$$LMOA = \{f \in VMOA : \lim_{|I| \to 0} \frac{\mu_f(S(I))}{|I|} \log(1/|I|) = 0\}.$$

Zhu [20] is a good reference for background on the spaces defined in this section.

The next lemma will be useful in Chapter 4 when showing T_g is not bounded below on $H^2, BMOA$, and the Bloch space.

Lemma 2.4. *If n is a positive integer, then $1 = \|z^n\|_2 \sim \|z^n\|_* \sim \|z^n\|_{\mathcal{B}}$, and the constants of comparison are independent of n.*

Proof. A straightforward calculation shows $\|z^n\|_2 = 1$ for all n. Checking the Bloch norm, we get $\|z^n\|_{\mathcal{B}} \sim \sup_{0<r<1} nr^{n-1}(1-r) = (1-\frac{1}{n})^{n-1} \to 1/e$ as $n \to \infty$. Finally, $1 = \|z^n\|_2 \lesssim \|z^n\|_* \lesssim \|z^n\|_\infty = 1$, by (2.1) and (2.2). □

On all the spaces mentioned, point evaluation is a bounded linear functional. For $f \in H^p (1 \le p < \infty), |f(z)| \le 2^{1/p}\|f\|_p(1-|z|)^{-1/p}$ (see [8, p. 36]). The norm of point evaluation at z in A^p_α is comparable to $(1-|z|)^{-(2+\alpha)/p}$ [20, Theorem 4.14]. In $\mathcal{B}$, the norm of point evaluation at z is comparable to $\log(2/(1-|z|))$, which we demonstrate in the next Proposition.

Proposition 2.5. *For $f \in \mathcal{B}$, let $\lambda^0_z f = f(z)$ denote point evaluation at z. Then*

$$\|\lambda^0_z\| \sim \log(2/(1-|z|)).$$

Proof. If $f \in \mathcal{B}$, then $|f'(z)| \le \|f\|_{\mathcal{B}}/(1-|z|)$ by the definition of $\mathcal{B}$ and the Maximum Modulus Principle. Integrating along a ray from the origin to $z = re^{i\theta} \in D$, we have

$$\begin{aligned} |f(z) - f(0)| &= \left|\int_0^z f'(w)\,dw\right| \\ &\le \int_0^r |f'(te^{i\theta})|\,dt \\ &\le \|f\|_{\mathcal{B}} \int_0^r 1/(1-t)\,dt \\ &= \|f\|_{\mathcal{B}} \log(1/(1-r)). \end{aligned}$$

Thus, $|f(z)| \le |f(0)| + \|f\|_{\mathcal{B}} \log(1/(1-|z|))$, and $\|\lambda^0_z\| \lesssim \log(2/(1-|z|))$. Recall that $\|\|_{\mathcal{B}}$ was defined as a semi-norm, so to account for $|z|$ near 0 we use $\log(2/(1-|z|))$.

For $a \in D$, define the test function

$$f_a(z) = \log(1/(1 - \overline{a}z)).$$

Then $|f_a'(z)| = |a|/|1 - \overline{a}z|$, and $f_a \in \mathcal{B}$ with $\|f_a\|_{\mathcal{B}} \leq 1$ for all a. Also,

$$\|\lambda_a^0\| \geq |f_a(a)| = \log(1/(1 - |a|^2)).$$

This shows $\log(2/(1 - |z|)) \lesssim \|\lambda_z^0\|$, hence the proposition is true. $\square$

Remark. This result generalizes to the α-Bloch spaces, with $\|\lambda_z^0\|_{\mathcal{B}_\alpha} \sim (1 - |z|^2)^{\alpha-1}$ for $\alpha > 1$. The proof is similar but the test functions must be adjusted.

An application of bounded point evaluation is the following well known result.

Proposition 2.6. *The following are equivalent:*

(i) *M_g is bounded on H^p for $1 \leq p \leq \infty$.*

(ii) *M_g is bounded on A_α^p for $1 \leq p < \infty, \alpha > -1$.*

(iii) *$g \in H^\infty$.*

Proof. If $g \in H^\infty, 1 \leq p < \infty$, then

$$\begin{aligned}\|M_g f\|_p = \|fg\|_p &= \sup_{0<r<1} \int_0^{2\pi} |f(re^{it})g(re^{it})|^p\, dt \\ &\leq \|g\|_\infty \sup_{0<r<1} \int_0^{2\pi} |f(re^{it})|^p\, dt \\ &= \|g\|_\infty \|f\|_p.\end{aligned}$$

If $p = \infty$, then $\|M_g f\|_\infty = \sup_{z \in D} |f(z)g(z)| \leq \|f\|_\infty \|g\|_\infty$. Conversely, if M_g is bounded on H^p $(1 \leq p \leq \infty)$, then $g \in H^\infty$ by Theorem 2.1. We have proved (i) and (iii) are equivalent. The proof that (ii) and (iii) are equivalent is similar. $\square$

2.3 Differentiation Isomorphisms

When studying T_g and S_g, it is useful to be able to compare the norm of a function to the norm of its derivative. For $p \geq 1$, $\alpha > -1$, the differentiation operator and its inverse, the indefinite integral, are isomorphisms between $A^p_\alpha/\mathbb{C}$ and $A^p_{\alpha+p}$, i.e.,

$$\|f\|_{A^p_\alpha} \sim |f(0)| + \|f'\|_{A^p_{\alpha+p}} \tag{2.3}$$

(see [20, Proposition 4.28]). Making the natural definition $A^2_{-1} = H^2$, (2.3) holds for $p = 2$, $\alpha = -1$ as well. For $f \in H^2$ with $f(0) = 0$, this is the well-known Littlewood-Paley identity,

$$\frac{1}{2\pi}\|f\|_2^2 = \frac{1}{\pi}\int_D 2|f'(z)|^2 \log\frac{1}{|z|}\, dA(z)$$

(see [10, Ch. IV, Sec. 3]). The relation (2.3) demonstrates a key connection between S_g and M_g via the differentiation operator, since $(S_g f)' = M_g f'$, and thus the following diagram is commutative.

$$\begin{array}{ccc} A^p_\alpha/\mathbb{C} & \xrightarrow{S_g} & A^p_\alpha/\mathbb{C} \\ {\scriptstyle f\mapsto f'}\big\downarrow & & \big\downarrow{\scriptstyle f\mapsto f'} \\ A^p_{\alpha+p} & \xrightarrow{M_g} & A^p_{\alpha+p} \end{array}$$

2.4 Boundedness of M_g

The multiplication operator M_g has been thoroughly studied, and conditions characterizing boundedness of M_g on the spaces mentioned are well known. The next theorem lists these results.

For a set $X \subset H(D)$, let

$$M[X] = \{g \in H(D) : M_g \text{ is bounded on } X\}.$$

Define $T[X] = \{g \in H(D) : T_g X \subset X\}$. Define $S[X]$ similarly.

Theorem 2.7. (i) $M[H^p] = H^\infty, 1 \leq p \leq \infty$.

(ii) $M[A^p_\alpha] = H^\infty, 1 \leq p < \infty, \alpha > 0$.

(iii) $M[\mathcal{B}] = H^\infty \cap \mathcal{B}_\ell$.

(iv) $M[BMOA] = H^\infty \cap LMOA$.

(v) *M_g is bounded on $\mathcal{D}$ if and only if $g \in H^\infty$ and the measure μ_g given by $d\mu_g(z) = |g'(z)|^2 dA(z)$ is a Carleson measure for $\mathcal{D}$.*

For (i) and (ii) see Proposition 2.6. The result for the Bloch space, (iii), is due originally to Jonathan Arazy [4]. The results (iv) and (v) for $BMOA$ and the Dirichlet space $\mathcal{D}$ are due to David Stegenga (see [17] and [18]). The Carleson measures for $\mathcal{D}$ were studied by Stegenga in [18], and characterized by a condition involving capacity.

2.5 Operators with Closed Range

A bounded operator T on a space X is said to be *bounded below* if there exists $C > 0$ such that $\|Tf\| \geq C\|f\|$ for all $f \in X$. It typically is the case for one-to-one operators on Banach spaces that boundedness below is equivalent to having closed range. The analogue of Theorem 2.9 for composition operators is found in Cowen and MacCluer [6]. We include the proof for T_g and S_g, essentially the same, for easy reference.

Lemma 2.8. *T_g is one-to-one for nonconstant g.*

Proof. Let $f_1, f_2 \in H(D)$. If $T_g f_1 = T_g f_2$, taking derivatives gives $f_1(z)g'(z) = f_2(z)g'(z)$. Thus $f_1(z) = f_2(z)$ except possibly at the (isolated) points where g' vanishes. Since f_1 and f_2 are analytic, $f_1 = f_2$. $\square$

When considering the property of being bounded below for S_g, we note that S_g maps any constant function to the 0 function. Thus, it is only useful to consider spaces of analytic functions modulo the constants.

Theorem 2.9. *Let Y be a Banach space of analytic functions on the disk, and let T_g and S_g be bounded on Y. For nonconstant g, T_g is bounded below on Y if and only if it has closed range. S_g is bounded below on $Y/\mathbb{C}$ if and only if it has closed range on $Y/\mathbb{C}$.*

Proof. Assume T_g is bounded below, i.e., there exists $\varepsilon > 0$ such that $\|T_g f\| \geq \varepsilon \|f\|$ for all f. Suppose $\{T_g f_n\}$ is a Cauchy sequence in the range of T_g. Since $\|f_n - f_m\| \lesssim \|T_g f_n - T_g f_m\|$, $\{f_n\}$ is also a Cauchy sequence. Letting $f = \lim f_n$, we have $T_g f_n \to T_g f$, showing $T_g f_n$ converges in the range of T_g. Hence the range is closed.

Conversely, assume $T_g : Y \to Y$ is closed range. Let $\{f_n\}$ be a sequence in Y such that $\|T_g f_n\| \to 0$. T_g is one-to-one by Lemma 2.8. Let the closed range of T_g be X. With the norm inherited from Y, X is a Banach space, and we can define the inverse $T_g^{-1} : X \to Y$. Suppose $\{x_n\}$ converges to $x = T_g h$ in X, and $T_g^{-1} x_n$ converges to y in Y. Applying T_g to $\{T_g^{-1} x_n\}$, this means x_n converges to $T_g y$. Hence $T_g y = T_g h$. Since T_g is one-to-one, $y = h = T_g^{-1} x$. By the Closed Graph Theorem, T_g^{-1} is continuous. Thus, $\|f_n\| = \|T_g^{-1}(T_g f_n)\| \to 0$, implying T_g is bounded below.

The same argument holds for S_g as well, but only on spaces modulo constants, since S_g is not one-to-one otherwise. $\square$

Theorem 2.10. *Let X be an infinite dimensional Banach space and $T : X \to X$ a bounded linear operator. If T is bounded below, then it is not compact.*

Proof. The special case when X is a Hilbert space is easy. Find an orthonormal sequence $\{u_n\} \subset X$. Assume T is bounded below, so there exists $\delta > 0$ such that $\|Tx\| \geq \delta \|x\|$ for all $x \in X$. Then, for $m \neq n$,

$$\|Tu_n - Tu_m\| = \|T(u_n - u_m)\| \geq \delta \|u_n - u_m\| = \delta\sqrt{2}.$$

Thus we have a uniformly separated sequence of points in the image of the closed unit ball of X under T. The separated sequence can have no convergent subsequence, showing T is not compact.

For a general Banach space X, we construct an analogous sequence. We show there is an infinite, uniformly separated sequence in the closed unit ball B of X. If this fails to be true, then for any $\varepsilon > 0$ there exists a finite set that is an ε cover of B. So suppose $\{u_1, u_2, ..., u_N\} \subset B$, and for all $u \in B$, there exists j, $1 \leq j \leq N$, such that $\|u - u_j\| < 1/2$. Let M be the span of $\{u_1, u_2, ..., u_N\}$. We will show that $M = X$, contradicting the assumption that X has infinite dimension. Let $y \in X$. Since M is finite dimensional, it is closed. If $y \notin M$, then $d = \inf_{m \in M} \|y - m\| > 0$. Let $m_0 \in M$ such that $d \leq \|y - m_0\| \leq 3d/2$. Let $y_0 = (y - m_0)/\|y - m_0\|$, so $y_0 \in B$. Then

$$\begin{aligned} \inf_{m \in M} \|y_0 - m\| &= \inf_{m \in M} \|\frac{y - m_0}{\|y - m_0\|} - m\| = \inf_{m \in M} \|\frac{y - m_0 - \|y - m_0\| m}{\|y - m_0\|}\| \\ &= \frac{1}{\|y - m_0\|} \inf_{m \in M} \|y - m\| \geq \frac{d}{3d/2} = 2/3. \end{aligned}$$

This violates the fact that the basis of M is a $1/2$ cover of B. We conclude that there exists an infinite sequence $\{u_n\} \subset B$ such that $m \neq n$ implies $\|u_n - u_m\| \geq 1/2$. As we saw in the case when X is a Hilbert space, if T is bounded below then it is not compact. $\square$

Chapter 3

Boundedness of T_g and S_g

3.1 Results of Aleman, Siskakis, Cima, and Zhao

Alexandru Aleman and Joseph Cima characterized boundedness and compactness of T_g on the Hardy spaces in [2] (Theorem 3.1 below). Recall from section 2.2 that the dual of H^1 is $BMOA$. The dual of the Bergman space A^1 is the Bloch space ([20, Theorem 5.1.4]). Hence, in light of duality, Theorem 3.2 is an analogue for the Bergman spaces of Theorem 3.1. Theorem 3.2 was proved by Aleman and Aristomenis Siskakis in [3]. Theorem 3.3 was established by Siskakis and Ruhan Zhao in [16].

Theorem 3.1. (Aleman and Cima [2]) *For $1 \leq p < \infty$, T_g is bounded [compact] on H^p if and only if $g \in BMOA\,[VMOA]$.*

Theorem 3.2. (Aleman and Siskakis [3]) *For $p \geq 1$, T_g is bounded [compact] on A^p if and only if $g \in \mathcal{B}\,[\mathcal{B}_0]$.*

Theorem 3.3. (Siskakis and Zhao [16]) *The following are equivalent.*

(i) *T_g is bounded on $BMOA$.*

(ii) T_g *is compact on* $BMOA$.

(iii) $g \in LMOA$.

3.2 The α-Bloch Spaces

The natural analogue of Theorem 3.3 is that T_g is bounded on $\mathcal{B}$ precisely when $g \in \mathcal{B}_\ell$. Theorem 3.4 extends this result to the α-Bloch spaces as well [12].

Theorem 3.4. *Let* $\alpha, \beta > 0$.

(a) *The operator* S_g *maps* $\mathcal{B}_\alpha$ *boundedly into* $\mathcal{B}_\beta$ *if and only if*

(i) $|g(z)| = O((1-|z|^2)^{\alpha-\beta})$ *as* $|z| \to 1^-$ $(\alpha \leq \beta)$.

(ii) $g = 0$ $(\alpha > \beta)$.

(b) *The operator* T_g *maps* $\mathcal{B}_\alpha$ *boundedly into* $\mathcal{B}_\beta$ *if and only if*

(i) $g \in \mathcal{B}_{\beta,\ell}$ $(\alpha = 1)$.

(ii) $g \in \mathcal{B}_{1-\alpha+\beta}$ $(\alpha > 1,\ 1-\alpha+\beta \geq 0)$.

(iii) g is constant $(\alpha > 1,\ 1-\alpha+\beta < 0)$.

(iv) $g \in \mathcal{B}_\beta$ $(\alpha < 1)$.

Proof. Recall from Proposition 2.5 that if $\alpha = 1$, we have $\|\lambda_z^0\|_{\mathcal{B}_\alpha} \sim \log \frac{1}{1-|z|}$ as $|z| \to 1$, where $\|\lambda_z^0\|_{\mathcal{B}_\alpha} = \sup_{\|f\|\leq 1} |f(z)|$ is the norm in $\mathcal{B}_\alpha$ of point evaluation at $z \in D$. If $\alpha > 1$, then $\|\lambda_z^0\|_{\mathcal{B}_\alpha} \sim (1-|z|^2)^{\alpha-1}$ as $|z| \to 1$.

Note that $\|\|_{\mathcal{B}_\alpha}$ are seminorms, which are adequate for showing boundedness of these operators. We consider conditions such that S_g maps $\mathcal{B}_\alpha$ into $\mathcal{B}_\beta$. Define a certain growth measurement of g by

$$A_t(g) = \sup_{z \in D}((1-|z|^2)^t |g(z)|), \qquad t \geq 0.$$

If $\beta \geq \alpha$ we have

$$\begin{aligned} \|S_g f\|_{\mathcal{B}_\beta} &= \sup_{z\in D}(|f'(z)g(z)|(1-|z|^2)^\beta) \\ &\leq \sup_{z\in D}(|f'(z)|(1-|z|^2)^\alpha) \sup_{z\in D}((1-|z|^2)^{\beta-\alpha}|g(z)|) \\ &= \|f\|_{\mathcal{B}_\beta} A_{\beta-\alpha}(g). \end{aligned}$$

Thus, S_g maps $\mathcal{B}_\alpha$ boundedly into $\mathcal{B}_\beta$ if $A_{\beta-\alpha}(g) < \infty$.

To show this condition is necessary, suppose S_g maps $\mathcal{B}_\alpha$ boundedly into $\mathcal{B}_\beta$ for $\beta \geq \alpha$. By Theorem 1,

$$|g(z)| \leq \|S_g\| \frac{\|\lambda_z^1\|_{\mathcal{B}_\beta}}{\|\lambda_z^1\|_{\mathcal{B}_\alpha}} \sim \|S_g\| \frac{(1-|z|^2)^{-\beta}}{(1-|z|^2)^{-\alpha}}.$$

Taking the supremum over $z \in D$, we get $A_{\beta-\alpha}(g) \lesssim \|S_g\|$. Hence S_g is bounded if and only if $A_{\beta-\alpha}(g) < \infty$. In particular, S_g is bounded on B_α if and only if $g \in H^\infty$, which is evident from Corollary 2.3. If S_g maps $\mathcal{B}_\alpha$ boundedly into $\mathcal{B}_\beta$, and $\beta < \alpha$, Theorem 2.2 implies that $|g(z)| \to 0$ as $|z| \to 1^-$. Since g is analytic, this means $g = 0$. This proves (a).

Consider conditions such that T_g maps $\mathcal{B}_\alpha$ into $\mathcal{B}_\beta$. By the Closed Graph Theorem, mapping $\mathcal{B}_\alpha$ into $\mathcal{B}_\beta$ is equivalent to T_g mapping $\mathcal{B}_\alpha$ boundedly into $\mathcal{B}_\beta$. In the case $\alpha = 1$, and $\|f\|_{\mathcal{B}} \neq 0$, we have $|f(z)| \lesssim \|f\|_{\mathcal{B}} \log(2/(1-|z|))$. Thus, if $g \in \mathcal{B}_{\beta,\ell}$, then

$$\begin{aligned} \|T_g f\|_{\mathcal{B}_\beta} &= \sup_{z\in\mathcal{D}} |f(z)||g'(z)|(1-|z|^2)^\beta \\ &\lesssim \sup_{z\in\mathcal{D}}\left(\|f\|_{\mathcal{B}} \log\frac{2}{1-|z|} |g'(z)|(1-|z|^2)^\beta \right) \\ &\leq \|f\|_{\mathcal{B}} \|g\|_{\mathcal{B}_{\beta,\ell}}, \end{aligned}$$

so T_g is bounded. Conversely, T_g being bounded implies, by Theorem 2.2,

$$|g'(z)| \leq \|T_g\| \frac{\|\lambda_z^1\|_{\mathcal{B}_\beta}}{\|\lambda_z^0\|_{\mathcal{B}}} \sim \|T_g\| \frac{(1-|z|)^{-\beta}}{\log \frac{2}{1-|z|}}.$$

Hence, T_g is bounded if and only if $g \in \mathcal{B}_{\beta,\ell}$.

In the case $\alpha > 1$, assume T_g maps $\mathcal{B}_\alpha$ into $\mathcal{B}_\beta$. Then, by Theorem 2.2,

$$|g'(z)| \leq \|T_g\| \frac{\|\lambda_z^1\|_{\mathcal{B}_\beta}}{\|\lambda_z^0\|_{\mathcal{B}_\alpha}} \sim (1-|z|^2)^{\alpha-\beta-1}.$$

If $1-\alpha+\beta > 0$, then $g \in \mathcal{B}_{1-\alpha+\beta}$. $1-\alpha+\beta = 0$ implies g is a function whose derivative is bounded. If $1-\alpha+\beta < 0$, then g is constant.

For $\alpha > 1$, $\|f\|_{\mathcal{B}} \neq 0$, we have $|f(z)| \lesssim \|f\|_{\mathcal{B}_\alpha}(1-|z|)^{1-\alpha}$. Thus,

$$\begin{aligned}
\|T_g f\|_{\mathcal{B}_\beta} &\lesssim \sup_{z\in\mathcal{D}} \big((\|f\|_{\mathcal{B}_\alpha}(1-|z|^2)^{1-\alpha})|g'(z)|(1-|z|^2)^\beta\big) \\
&\leq \|g\|_{\mathcal{B}_\beta} + \|f\|_{\mathcal{B}_\alpha}\|g\|_{\mathcal{B}_{1-\alpha+\beta}}.
\end{aligned}$$

In the case $\alpha < 1$, $\mathcal{B}_\alpha$ is a Lipschitz class space (see [8]), a subspace of H^∞. Evidently T_g is bounded from $\mathcal{B}_\alpha$ to $\mathcal{B}_\beta$ if and only if $g \in \mathcal{B}_\beta$. $\square$

3.3 Splitting the Multiplier Condition

We compare the three operators M_g, T_g, and S_g. Recall $M[X] = \{g \in H(D) : M_g \text{ is bounded on } X\}$. Define $T[X]$ and $S[X]$ similarly. The Dirichlet space provides a nice example of how the condition for boundedness of M_g may split into the conditions for S_g and T_g.

A function $f \in H(D)$ is in the Dirichlet space $\mathcal{D}$ provided

$$\|f\|_{\mathcal{D}} = \int_D |f'(z)| \, dA(z) < \infty,$$

where A is Lebesgue area measure on D. A complex measure μ is a *Carleson measure for* $\mathcal{D}$ if, for all $f \in \mathcal{D}$, $\int_D |f|^2 \, d\mu \leq \|f\|_{\mathcal{D}}^2$. Carleson measures for $\mathcal{D}$ were characterized by Stegenga in [18]. Stegenga uses the result to characterize the pointwise multipliers of the Dirichlet space, which depend on two conditions. One is boundedness of the symbol of the multiplier. The other condition is a capacitary condition on the measure μ in Theorem 3.5 (ii), the same condition characterizing the symbols g for which T_g is bounded. This is not surprising since the two conditions come from the two terms in the product rule, which also gives us $M_g \sim T_g + S_g$.

Theorem 3.5. (i) *S_g is bounded on the Dirichlet space $\mathcal{D}$ if and only if $g \in H^\infty$.*

(ii) *T_g is bounded on the Dirichlet space $\mathcal{D}$ if and only if μ is a Carleson measure for $\mathcal{D}$, where $d\mu(z) = |g'(z)|^2 dA(z)$.*

Proof.

S_g is bounded on $\mathcal{D}$ if and only if there exists $C > 0$ such that

$$\| S_g f \|_{\mathcal{D}}^2 = \int_{\mathcal{D}} |f'(z)|^2 |g(z)|^2 \, dA(z) \leq C \, \| f \|_{\mathcal{D}}^2 \, .$$

Clearly $g \in H^\infty$ implies S_g is bounded. For the converse, note that a function f is in $\mathcal{D}$ if and only if $f' \in A_0^2$, so point evaluation of the derivative is bounded on $\mathcal{D}$. Thus, Corollary 2.3 applies, proving (i).

T_g is bounded on $\mathcal{D}$ if and only if there exists $C > 0$ such that

$$\| T_g f \|_{\mathcal{D}}^2 = \int_{\mathcal{D}} |f(z)|^2 |g'(z)|^2 \, dA(z) \leq C \| f \|_{\mathcal{D}}^2 .$$

This is precisely the statement in (ii). $\square$

The interaction of M_g, S_g, and T_g imitates the situation in the Bloch space and in $BMOA$. On $\mathcal{D}, BMOA$, and $\mathcal{B}$, M_g is bounded only if S_g and T_g are both bounded, yet it is possible for S_g or T_g to be unbounded while M_g is bounded. The two conditions given for multipliers of $\mathcal{B}$ in Theorem 2.7 are the same as the conditions for S_g and T_g to be bounded, namely that $g \in H^\infty$ and that $g \in \mathcal{B}_\ell$. On $BMOA$, S_g is bounded if and only if $g \in H^\infty$, T_g is bounded if and only if $g \in LMOA$, and M_g is bounded when $g \in H^\infty \cap LMOA$. The following table summarizes these results. ($Y = T[\mathcal{D}]$. See Theorem 3.5.)

X	$M[X]$	$S[X]$	$T[X]$
$\mathcal{D}$	$H^\infty \cap Y$	H^∞	Y
$BMOA$	$H^\infty \cap LMOA$	H^∞	$LMOA$
$\mathcal{B}$	$H^\infty \cap \mathcal{B}_\ell$	H^∞	$\mathcal{B}_\ell$
H^p	H^∞	H^∞	$BMOA$
A^p	H^∞	H^∞	$\mathcal{B}$
H^∞	H^∞	?	?

In the Hardy and Bergman spaces, the condition characterizing multipliers is simply the condition for S_g to be bounded, but the condition for boundedness of T_g is weaker, i.e., $S[X] \subset T[X]$. Thus, in all these spaces we have the phenomenon that boundedness of the multiplication operator M_g is equivalent to boundedness of both S_g and T_g. As we will see in Section 3.4 below, this phenomenon fails for the operators acting on H^∞. The question marks in the table represent unsolved problems, but some discussion and partial results will be presented.

3.4 Boundedness of T_g and S_g on H^∞

It is trivial that $M[H^\infty] = H^\infty$, i.e., the multipliers of H^∞ are precisely the functions in H^∞ themselves. Such is not the case for T_g and S_g, and characterizing boundedness of these operators on H^∞ is an open problem. We investigate this problem. The following proposition gives a necessary condition.

Proposition 3.6. $T[H^\infty] = S[H^\infty] \subseteq H^\infty$.

Proof. From Theorem 2.2, we see that $S[H^\infty] \subseteq H^\infty$, i.e., if S_g is bounded on H^∞, then $g \in H^\infty$. Hence M_g is bounded as well. By the product rule, (1.1), this implies T_g is also bounded. Thus, S_g is bounded implies T_g is bounded, or $S[H^\infty] \subseteq T[H^\infty]$. Letting $1 \in H^\infty$ denote the constant function, we have $T_g 1 = g$. If T_g is bounded with norm $\|T_g\|$, then

$$\|g\|_\infty = \|T_g 1\|_\infty \leq \|T_g\|.$$

Thus $T[H^\infty] \subseteq H^\infty$. If T_g is bounded then $g \in H^\infty$ and M_g is bounded, so S_g is bounded by (1.1), i.e., $T[H^\infty] \subseteq S[H^\infty]$. Hence the result holds. $\square$

We show that the inclusion in Proposition 3.6 is proper, i.e., $g \in H^\infty$ is not sufficient for T_g to be bounded. The following counterexample demonstrates this.

For $w, z \in D$, let $\rho(z, w) = \frac{|w-z|}{|1-\overline{w}z|}$ denote the pseudohyperbolic metric on D, and for $0 < r < 1$ let $D(w, r) = \{z \in D : \rho(z, w) < r\}$. We can find a sequence $\{a_n\}$ such that, for the Blaschke product B with zeros $\{a_n\}$, T_B is unbounded on H^∞. Fix a small $\varepsilon > 0$. We will choose $\{a_n\}$ such that $0 < a_n < a_{n+1} < 1$ for all n, with corresponding factors $\sigma_n(x) = \frac{a_n - x}{1 - a_n x}$, so $B = \prod \sigma_n$ is real-valued on the unit interval. For each n define $B_n = B/\sigma_n$. Also, choose the a_n to be highly separated in pseudohyperbolic distance; that is,

$$|B_n(x)| > 1 - \varepsilon \text{ for } x \in I_n,$$

where $I_n = D(a_n, 1/2) \cap \mathbb{R}$. Let x_n and y_n be the endpoints of I_n, so $\sigma_n(x_n) = 1/2$ and $\sigma_n(y_n) = -1/2$.

$$\begin{aligned} |B(x_n) - B(y_n)| &= |B_n(x_n)\sigma_n(x_n) - B_n(x_n)\sigma_n(y_n) + B_n(x_n)\sigma_n(y_n) - B_n(y_n)\sigma_n(y_n)| \\ &\geq |B_n(x_n)\sigma_n(x_n) - B_n(x_n)\sigma_n(y_n)| - |B_n(x_n)\sigma_n(y_n) - B_n(y_n)\sigma_n(y_n)| \\ &= |B_n(x_n)||\sigma_n(x_n) - \sigma_n(y_n)| - |\sigma_n(y_n)||B_n(x_n) - B_n(y_n)| \\ &= (1-\varepsilon)(1) - (1/2)\varepsilon \sim 1. \end{aligned}$$

Hence

$$\int_{I_n} |B'(x)|\, dx \sim 1$$

for all n.

We wish to estimate the zeros of B'. For $a_n < x < a_{n+1}$, n being odd implies $B(x) < 0$, and $B(x) > 0$ for even n. Let J_n be the interval between I_n and I_{n+1}, so $J_n = (y_n, x_{n+1})$. For odd n, $B(y_n)$ and $B(x_{n+1})$ are near $-1/2$, and B achieves a minimum value near -1 at a point approximately equidistant in pseudohyperbolic distance to a_n and a_{n+1}.[1]This point is pseudohyperbolically separated from I_n. Similarly, for even n B achieves a maximum, a zero of B', at a point between a_n and a_{n+1} separated from I_n.

[1]To be more precise, assume we have chosen ε small enough and $\{a_n\}$ separated enough that in

For each n there is only one zero of B' on the real line between a_n and a_{n+1}. The number of zeros in the disk is $n-1$ for the derivative of $\prod_{j=1}^{n} \sigma_j$ by the Riemann-Hurwitz formula, and Hurwitz's theorem tells us no other zeros arise in the limit function. Denote this zero d_n.

Letting $-f$ denote the Blaschke product with zero sequence $\{d_n\}$, we get $f(x)B'(x) \geq 0$ for $0 < x < 1$. Note that f is an interpolating Blaschke product, and there exists $\delta > 0$ such that for all n $f(x) \geq \delta$ for $x \in I_n$.

Then

$$\begin{aligned}
\lim_{r\to 1} T_B f(r) &= \lim_{r\to 1} \int_0^r B'(x) f(x)\, dx \\
&= \sum_n \int_{I_n} |B'(x)||f(x)|\, dx + \sum_n \int_{J_n} |B'(x)||f(x)|\, dx \\
&\gtrsim \sum_n \delta = \infty.
\end{aligned}$$

Hence T_B is not bounded on H^∞.

3.5 Future Work

The weakest sufficient condition we know for characterizing $T[H^\infty]$ is the one in Theorem 3.7, uniform boundedness of the radial variation of the symbol g. The

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a pseudohyperbolic neighborhood of $1/4$ around the endpoints of I_n, $|B_n| > 7/8$, and for $x \in J_n$,

$$\frac{|B(x)|}{|\sigma_n(x)\sigma_{n+1}(x)|} > 7/8.$$

Then for odd n, $x \in D(y_n, 1/4)$, we have $(7/8)(-1/2) > B(x) > -1/2$, which also holds for $x \in D(x_{n+1}, 1/4)$. For $x \in J_n$ such that $\rho(a_n, x) > 7/8$ and $\rho(a_{n+1}, x) > 7/8$, (having chosen $\{a_n\}$ to ensure such x exists) we have $|B(x)| > (7/8)^3 > 1/2$. As B is continuous, the Mean Value Theorem gives us a zero of B' in J_n. Although we have not proved this zero is pseudohyperbolically separated from I_n, it is separated from a neighborhood of a_n of radius $1/4$, which is enough to draw our conclusion.

strongest necessary condition we have proven is that $g \in H^\infty$, although we know this is not sufficient. The *radial variation* of $f \in H(D)$ at $\theta \in \partial D$ is

$$V(f,\theta) = \int_0^1 |f'(te^{i\theta})|\, dt.$$

In this section we examine the problem and the condition of having bounded radial variation. Finally we give the weakest sufficient condition we have for compactness of T_g in Theorem 3.12.

Theorem 3.7. *The following condition on g implies T_g is bounded on H^∞:*

$$\text{There exists } M > 0 \text{ such that for all } \theta \in \partial D, V(g,\theta) < M. \tag{3.1}$$

Proof. If (3.1) holds, then

$$\begin{aligned}
\|T_g f\|_\infty &= \sup_{z\in D} \left|\int_0^z f(w)g'(w)\, dw\right| \\
&= \sup_\theta \left|\int_0^1 f(te^{i\theta})g'(te^{i\theta})\, dt\right| \\
&\le \sup_\theta \int_0^1 |f(te^{i\theta})g'(te^{i\theta})|\, dt \le \|f\|_\infty M.
\end{aligned}$$

□

The following proposition is the Fejér-Riesz inequality. For a proof, see [8, 3.13].

Proposition 3.8. (Fejér-Riesz) *If $f \in H^p \quad (1 \le p < \infty)$, then the integral of $|f|^p$ along the real interval $-1 < x < 1$ converges, and*

$$\int_{-1}^1 |f(x)|^p\, dx \le \frac{1}{2}\|f\|_p^p.$$

By Theorem 3.7 and the Fejér-Riesz inequality, $g' \in H^1$ implies T_g is bounded on H^∞, with norm no greater than $\|g'\|_{H^1}/2$. However, condition 3.1 does not imply $g' \in H^1$, as we will see in Theorem 3.11. We will use a pair of theorems from univalent function theory.

For $E \subset \mathbb{C}$, $\Lambda(E)$ denotes the linear measure of E.

$$\Lambda(E) = \lim_{\varepsilon \to 0} \inf_{d_k < \varepsilon} \sum_k d_k,$$

where the infimum ranges over countable covers of E by discs D_k of diameter d_k.

Theorem 3.9. [14, Theorem 10.11] *If $f(z)$ is analytic and univalent in D then*

$$f' \in H^1 \Leftrightarrow \Lambda(\partial f(D)) < \infty.$$

The next Theorem says that hyperbolic geodesics are roughly no longer than euclidean geodesics. It is attributed to F. W. Gehring and W. K. Hayman.

Theorem 3.10. [14, Theorem 10.9] *Let $f(z)$ be analytic and univalent in D. If $C \subset D$ is a Jordan arc from 0 to $e^{i\theta}$ then*

$$V(f,\theta) \leq K\, l(f(C)),$$

where K is an absolute constant, and $l(f(C))$ is the arc length of $f(C)$.

Theorem 3.11. *There exists a univalent function $f \in H^\infty$ such that* (3.1) *holds but $f' \notin H^1$.*

Proof. Construct a starlike region $G \subset D$ with a boundary of infinite length by removing countably many slits from D. The Riemann map f from D to G is univalent and fails to satisfy $f' \in H^1$ by Theorem 3.9. However, the image under f of any radius from 0 to $e^{i\theta}$ is bounded in length by K from Theorem 3.10. This shows that the condition $g' \in H^1$ is strictly stronger than condition (3.1). $\square$

The next result pertains to compactness of T_g.

Theorem 3.12. *The following condition implies compactness of T_g on H^∞:*

$$\textit{For all } \varepsilon > 0, \textit{ there exists } r < 1 \textit{ such that } \int_r^1 |g'(te^{i\theta})|\,dt < \varepsilon \textit{ for all } \theta \in \partial D. \quad (3.2)$$

Proof. Let X and Y be elements of the following set of Banach spaces: $\{H^p\ (1 \le p \le \infty), A^p\ (1 \le p < \infty), \mathcal{B}, BMOA, \mathcal{D}\}$. Let $\{f_n\}$ be a sequence of functions in X such that $\| f_n \|_X \le 1$ for all n. Standard methods show the following condition implies compactness of T_g from X to Y:

$$f_n \to 0 \text{ uniformly on compact subsets of } D \text{ implies } \quad \| T_g f_n \|_Y \to 0.$$

Let $\varepsilon > 0$, and assume (3.2) holds. Choose N such that $n > N$ implies $|f_n(z)| < \varepsilon$ for $|z| < r$, where r satisfies (3.2). Note that (3.1) holds, since there exists M such that $|g'(z)| < M$ on $\{z : |z| \le r\}$.

Then, for $z \in D$,

$$\begin{aligned}
\|T_g f_n\|_\infty &\le \sup_\theta \int_0^1 |f_n(te^{i\theta}) g'(te^{i\theta})|\,dt \\
&= \sup_\theta \int_0^r |f_n(te^{i\theta}) g'(te^{i\theta})|\,dt \ + \int_r^1 |f_n(te^{i\theta}) g'(te^{i\theta})|\,dt \\
&\le \varepsilon M + \varepsilon.
\end{aligned}$$

Hence $\|T_g f_n\|_\infty \to 0$ as $n \to \infty$. $\square$

Chapter 4

Results on Closed Range Operators

4.1 When T_g Is Bounded Below

We will show that T_g is never bounded below on H^2, $\mathcal{B}$, nor $BMOA$. The sequence $\{z^n\}$ demonstrates the result in each space, since the functions z^n have norm comparable to 1, independent of n (Lemma 2.4).

Theorem 4.1. *T_g is never bounded below on H^2, $\mathcal{B}$, nor $BMOA$.*

Proof. Let $f_n(z) = z^n$. For H^2,

$$\lim_{n\to\infty} \|T_g f_n\|_2^2 \sim \lim_{n\to\infty} \int_D |z^n|^2 |g'(z)|^2 (1-|z|^2)\, dA(z).$$

We assume T_g is bounded, so $g \in BMOA$ by the result of Aleman and Siskakis [3]. Thus μ_g is a Carleson measure, allowing us to bring the limit inside the integral by the Dominated Convergence Theorem.

$$\lim_{n\to\infty} \|T_g f_n\|_2^2 = \int_D \lim_{n\to\infty} |z^n|^2 |g'(z)|^2 (1-|z|^2)\, dA(z) = 0.$$

Since $\|f_n\|_2 = 1$ for all n, T_g is not bounded below.

If T_g is bounded on $\mathcal{B}$, then, by Theorem 2.2, $|g'(z)|(1-|z|) = O(1/\log(1/(1-|z|)))$ as $|z| \to 1$.

$$\|T_g f_n\|_{\mathcal{B}} = \sup_{z \in D} |z^n||g'(z)|(1-|z|) \lesssim \sup_{0 \le r < 1} r^n \frac{1}{\log(2/(1-r))}.$$

Given $\varepsilon > 0$, there exists $\delta < 1$ such that $1/\log(2/(1-r)) < \varepsilon$ for $\delta < r < 1$. For large n, $r^n < \varepsilon$ for $0 < r < \delta$. Thus, $\lim_{n \to \infty} \|T_g f_n\|_{\mathcal{B}} = 0$, and Lemma 2.4 implies T_g is not bounded below on $\mathcal{B}$.

Siskakis and Zhao [16] proved that if T_g is bounded on $BMOA$ then $g \in LMOA$.

$$\lim_{n \to \infty} \|T_g f_n\|_*^2 \sim \lim_{n \to \infty} \sup_I \frac{1}{|I|} \int_{S(I)} |z^n|^2 |g'(z)|^2 (1-|z|^2)\, dA(z).$$

Let I be an arc in ∂D, and let $\varepsilon > 0$. Since $g \in VMOA$, there exists $\delta > 0$ such that

$$\frac{1}{|J|} \int_{S(J)} |g'(z)|^2 (1-|z|^2)\, dA(z) < \varepsilon \text{ whenever } |J| < \delta.$$

If $|I| > \delta$, divide I into K disjoint intervals of length approximately δ, so

$$I = \cup_{i=1}^K J_i, \quad \delta/2 < |J_i| < \delta \text{ for all } i, \text{ and } \delta K \sim |I|.$$

Let $S_\delta(I) = S(I) - \cup_i S(J_i)$. For large n, $(1-\delta/2)^{2n} \le \varepsilon |I|$, and to estimate the integral over $S_\delta(I)$ we use the fact that μ_g is a Carleson measure.

$$\begin{aligned}
\frac{1}{|I|}\int_{S(I)} |z^n|^2 |g'(z)|^2 (1-|z|^2)\, dA(z) &= \frac{1}{|I|}\int_{S_\delta(I)} |z^n|^2 |g'(z)|^2 (1-|z|^2)\, dA(z) \\
&+ \frac{1}{|I|}\sum_{i=1}^{K}\int_{S(J_i)} |z^n|^2 |g'(z)|^2 (1-|z|^2)\, dA(z) \\
&\leq \frac{1}{|I|}(1-\delta/2)^{2n}\, C\|g\|_*^2 + \frac{1}{|I|}K\delta\varepsilon \lesssim \varepsilon
\end{aligned}$$

for large n. Hence $\lim_{n\to\infty}\|T_g f_n\|_* = 0$ and T_g is not bounded below on $BMOA$. $\square$

In contrast to Theorem 4.1, T_g can be bounded below on weighted Bergman spaces. We state the result here, but the key is Proposition 4.3, proved afterward.

Theorem 4.2. *Let $1 \leq p < \infty$, $\alpha > -1$. T_g is bounded below on A^p_α if and only if there exist $c > 0$ and $\delta > 0$ such that*

$$|\{z \in D : |g'(z)|(1-|z|^2) > c\} \cap S(I)| > \delta |I|^2.$$

Proof. We must assume T_g is bounded on A^p_α. By Theorem 2.2, $g \in \mathcal{B}$. (That this is also sufficient for T_g to be bounded on A^p_0 is in [2].) T_g is bounded below on A^p_α if and only if

$$\|T_g f\|^p_{A^p_\alpha} \sim \int_D |f(z)|^p |g'(z)|^p (1-|z|^2)^{\alpha+p}\, dA(z) \gtrsim \|f\|^p_{A^p_\alpha}. \tag{4.1}$$

By Proposition 4.3, (4.1) holds if and only if there exist $c > 0$ and $\delta > 0$ such that

$$|\{z \in D : |g'(z)|^p(1-|z|^2)^p > c\} \cap S(I)| > \delta |I|^2 \tag{4.2}$$

for all arcs $I \subseteq \partial D$. If (4.2) holds for some $p \geq 1$, it holds for $1 \leq p < \infty$, with an adjustment of the constant c. $\square$

The proof of [15, Proposition 5.4] shows this result is nonvacuous. Ramey and Ullrich construct a Bloch function g such that $|g'(z)|(1-|z|) > c_0$ if $1 - q^{-(k+1/2)} \leq |z| \leq 1 - q^{-(k+1)}$, for some $c_0 > 0$, q some large positive integer, and $k = 1, 2, \ldots.$ Given a Carleson square $S(I)$, let k_I be the least positive integer such that $q^{-k_I+1/2} \leq |I|$. The annulus $E = \{z : 1 - q^{-(k_I+1/2)} \leq |z| \leq 1 - q^{-(k_I+1)}\}$ intersects $S(I)$, and

$$|E \cap S(I)| \sim |I|((1 - q^{-(k_I+1)}) - (1 - q^{-(k_I+1/2)})) = |I|\frac{q^{1/2}-1}{q^{k_I+1}} \geq \frac{(q^{1/2}-1)}{q^{3/2}}|I|^2.$$

Setting $c = c_0$ and $\delta \sim 1/q$ show Theorem 4.2 holds for this example of g, and T_g is bounded below on A^p_α.

4.2 Boundedness Below of S_g

The operator S_g can clearly be bounded below, since $g(z) = 1$ gives the identity operator. A result due to Daniel Luecking (see [6, Theorem 3.34]) leads to a characterization of functions for which S_g is bounded below on $H^2_0 := H^2/\mathbb{C}$ and $A^p_\alpha/\mathbb{C}$. We state a reformulation useful to our purposes here.

Proposition 4.3. (Luecking) *Let τ be a bounded, nonnegative, measurable function on D. Let $G_c = \{z \in D : \tau(z) > c\}$, $1 \leq p < \infty$, and $\alpha > -1$. There exists $C > 0$ such that the inequality*

$$\int_D |f(z)|^p \tau(z)(1-|z|)^\alpha \, dA(z) \geq C \int_D |f(z)|^p (1-|z|)^\alpha \, dA(z)$$

holds if and only if there exist $\delta > 0$ and $c > 0$ such that $|G_c \cap S(I)| \geq \delta |I|^2$ for every *interval $I \subset \partial D$.*

The proof is omitted. Using the Littlewood-Paley identity we get the following:

Corollary 4.4. *S_g is bounded below on H_0^2 if and only if there exist $c > 0$ and $\delta > 0$ such that $|G_c \cap S(I)| \geq \delta |I|^2$, where $G_c = \{z \in D : |g(z)| > c\}$.*

We use Corollary 4.4 to construct an example when S_g is not bounded below on H_0^2, and compare M_g on H^2 to S_g on H_0^2. If $g(z)$ is the singular inner function $\exp(\frac{z+1}{z-1})$, S_g is not bounded below on H_0^2. To see this, fix $c \in (0,1)$. G_c is the complement in D of a horodisk, a disk tangent to the unit circle, with radius $r = \frac{\log c+1}{2(\log c-1)}$ and center $1-r$. Choosing a sequence of intervals $I_n \subset \partial D$ such that 1 is the center of I_n and $|I_n| \to 0$ as $n \to \infty$, we see

$$\frac{|G_c \cap S(I_n)|}{|I_n|^2} \to 0 \text{ as } n \to \infty,$$

meaning S_g is not bounded below on H_0^2.

M_g is bounded below on H^2 if and only if the radial limit function of $g \in H^\infty$ is essentially bounded away from 0 on ∂D (a special case of weighted composition operators; see [11]). Theorem 4.6 will show this is weaker than the condition for S_g to be bounded below on H_0^2. The example above of a singular inner function then shows it is strictly weaker. To prove Theorem 4.6 we use a lemma which allows us to estimate an analytic function inside the disk by its values on the boundary.

For any arc $I \subseteq \partial D$ and $0 < r < 2\pi/|I|$, rI will denote the arc with the same center as I and length $r|I|$. We define the upper Carleson rectangle

$$S_\varepsilon(I) = \{re^{it} : 1 - |I| < r < (1 - \varepsilon|I|), e^{it} \in I\}, \text{ and } S^+(I) = S_{1/2}(I).$$

Lemma 4.5. *Given ε, $0 < \varepsilon < 1$, and a point $e^{i\theta}$ such that $|g^*(e^{i\theta})| < \varepsilon$, there exists an arc $I \subset \partial D$ such that $|g(z)| < \varepsilon$ for $z \in S_\varepsilon(I)$.*

Proof. We can choose α close enough to 1 so that $S_\varepsilon(I) \subset \Gamma_\alpha(e^{i\theta})$ for all I centered at $e^{i\theta}$ with, say, $|I| < 1/4$. If $|g^*(e^{i\theta})| < \varepsilon$, there exists $\delta > 0$ such that

$$z \in \Gamma_\alpha(e^{i\theta}), |z - e^{i\theta}| < \delta \text{ imply } |g(z)| < \varepsilon.$$

Choosing I such that $S(I)$ is contained in a δ-neighborhood of $e^{i\theta}$ finishes the proof. $\square$

Theorem 4.6. *If S_g is bounded below on H_0^2, then M_g is bounded below on H^2.*

Proof. Assume M_g is not bounded below on H^2. Let $\varepsilon > 0$. The radial limit function of g equals g^* almost everywhere, so there exists a point $e^{i\theta}$ such that $|g^*(e^{i\theta})| < \varepsilon$. By Lemma 4.5, there exists $S(I)$ such that $|\{z : |g(z)| \geq \varepsilon\} \cap S(I)| \leq \varepsilon |I|$. Since ε was arbitrary, this violates the condition in Proposition 4.3. $\square$

4.2.1 S_g on the Bloch Space

We now characterize the symbols g which make S_g bounded below on the Bloch space. It turns out to be a common condition appearing in a few different forms in the literature. The condition appears in characterizing M_g on A_0^2 in McDonald and Sundberg [13]. Our main result is equivalence of (i)-(iii) in Theorem 4.7, and we give references with brief explanations for (iv)-(vi).

Theorem 4.7. *The following are equivalent for $g \in H^\infty$:*

(i) *$g = BF$ for a finite product B of interpolating Blaschke products and F such that $F, 1/F \in H^\infty$.*

(ii) S_g *is bounded below on* $\mathcal{B}/\mathbb{C}$.

(iii) *There exist* $r < 1$ *and* $\eta > 0$ *such that for all* $a \in D$,

$$\sup_{z \in D(a,r)} |g(z)| > \eta.$$

(iv) S_g *is bounded below on* H_0^2.

(v) M_g *is bounded below on* A_α^p *for* $\alpha > -1$.

(vi) S_g *is bounded below on* $A_\alpha^p/\mathbb{C}$ *for* $\alpha > -1$.

Proof. (i) $\Rightarrow$ (ii): Note that $S_{g_1 g_2} = S_{g_1} S_{g_2}$ for any g_1, g_2. It follows that if S_{g_1} and S_{g_2} are bounded below then $S_{g_1 g_2}$ is also bounded below. We will show that S_F and S_B are bounded below, implying the result for S_g.

If S_g is bounded on $\mathcal{B}$, then $g \in H^{infty}$ by Corollary 2.3. If F, $1/F \in H^\infty$, then

$$\|S_F f\| = \sup_{z \in D} |F(z)||f'(z)|(1 - |z|^2) \geq (1/\|1/F\|_\infty)\|f\|_{\mathcal{B}}.$$

Hence S_F is bounded below.

By virtue of the fact beginning this proof, we may assume B is a single interpolating Blaschke product without loss of generality. Let $\{w_n\}$ be the zero sequence of B, so

$$B(z) = e^{i\varphi} \prod_n \frac{w_n - z}{1 - \overline{w}_n z}.$$

Denote the pseudohyperbolic metric

$$\rho(z, w) = \frac{|w - z|}{|1 - \overline{w} z|}, \text{ for any } z, w \in D.$$

For the pseudohyperbolic disk of radius $d > 0$ and center $w \in D$, we use the notation

$$D(w, d) = \{z \in D : \rho(z, w) < d\}.$$

Let B_j be B without its jth zero, i.e., $B_j(z) = \frac{1-\overline{w}_j z}{w_j - z} B(z)$. Since B is interpolating, there exist $\delta > 0$ and $r > 0$ such that, for all j, $|B_j(z)| > \delta$ whenever $z \in D(w_j, r)$. In particular, the sequence $\{w_n\}$ is separated, so shrinking r if necessary, we may assume

$$\inf_{j \neq k} \rho(w_k, w_j) > 2r.$$

We compare $\|f\|$ to $\|S_B f\| = \sup_{z \in D} |B(z)||f'(z)|(1 - |z|^2)$. Let $a \in D$ be a point where the supremum defining the norm of f is almost achieved, say, $|f'(a)|(1-|a|^2) > \|f\|/2$.

Consider the pseudohyperbolic disk $D(a, r)$. Inside $D(a, r)$ there may be at most one zero of B, say w_k. We examine three cases depending on the location and existence of w_k.

If $r/2 \leq \rho(w_k, a) < r$, then

$$|B(a)| = \frac{|w_k - a|}{|1 - \overline{w}_k a|} |B_k(a)| > (r/2)\delta.$$

Thus we would have

$$\|S_B f\| \geq |B(a)||f'(a)|(1 - |a|^2) > (r/2)\delta \|f\|/2,$$

and S_g would be bounded below.

On the other hand, suppose $\rho(w_k, a) < r/2$. Consider the disk $D(w_k, r/2)$, which is contained in $D(a,r)$. The expression $1-|z|^2$ is roughly constant on a pseudohyperbolic disk, i.e.,

$$\sup_{z\in D(a,r)}(1-|z|^2) > C_r(1-|a|^2) \text{ for some } C_r > 0.$$

C_r does not depend on a, and is near 1 for small r. By the maximum principle for f', there exists a point $z_a \in \partial D(w_k, r/2)$ where

$$|f'(z_a)|(1-|z_a|^2) > |f'(a)|C_r(1-|a|^2) > C_r\|f\|/2.$$

(Since $\rho(w_k,a) < r/2$ and $\rho(z_a,w_k) = r/2$, we have $\rho(z_a,a) < r$.) This shows that S_g is bounded below, for

$$\begin{aligned}\|S_B f\| &\geq |B(z_a)||f'(z_a)|(1-|z_a|^2)\\ &> \rho(w_k,z_a)|B_k(z_a)|C_r\|f\|/2\\ &> (r/2)\delta C_r\|f\|/2.\end{aligned}$$

Finally, suppose no such w_k exists. Then the function $((a-z)/(1-\overline{a}z))B(z)$ is also an interpolating Blaschke product, and the previous case applies with $w_k = a$.

(ii) $\Rightarrow$ (iii): Assume (iii) fails. Given $\varepsilon > 0$, choose r near 1 so that $1-r^2 < \varepsilon$, and choose $a \in D$ such that $|g(z)| < \varepsilon$ for all $z \in D(a,r)$. Consider the test function $f_a(z) = (a-z)/(1-\overline{a}z)$. By a well-known identity,

$$(1-|z|^2)|f_a'(z)| = 1-(\rho(a,z))^2.$$

Thus $f_a \in \mathcal{B}$ with $\|f_a\| = 1$ for all $a \in D$. (The seminorm is 1, but the true norm is between 1 and 2 for all a.) By supposition on g,

$$
\begin{aligned}
\|S_g f_a\| &= \sup_{z \in D} |g(z)||f_a'(z)|(1-|z|^2) \\
&= \max\left\{ \sup_{z \in D(a,r)} |g(z)||f_a'(z)|(1-|z|^2), \sup_{z \in D \setminus D(a,r)} |g(z)||f_a'(z)|(1-|z|^2) \right\} \\
&\leq \max\left\{ \sup_{z \in D(a,r)} |g(z)|\|f_a\|, \sup_{z \in D \setminus D(a,r)} |g(z)|(1-r^2) \right\} \\
&< \max\{\varepsilon, \|g\|_\infty \varepsilon\} \leq \varepsilon(\|g\|_\infty + 1)
\end{aligned}
$$

Since $\|f_a\| = 1$ and ε was arbitrary, S_g is not bounded below.

(iii) $\Rightarrow$ (i): Assuming (iii) holds, we first rule out the possibility that g has a singular inner factor. We factor $g = BI_gO_g$ where B is a Blaschke product, I_g a singular inner function, and O_g an outer function. Let ν be the measure on ∂D determining I_g, so

$$I_g(z) = \exp\left(-\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\nu(\theta)\right).$$

Let $\varepsilon > 0$. For any $\alpha > 1$ and for ν-almost all θ, there exists $\delta > 0$ such that

$$z \in \Gamma_\alpha(e^{i\theta}), |z - e^{i\theta}| < \delta \text{ imply } |I_g(z)| < \varepsilon. \tag{4.3}$$

This is [10, Theorem II.6.2]. δ may depend on θ and α, but for nontrivial ν there exists some θ where (4.3) holds. Given $r < 1$, choose $\alpha < 1$ such that, for every a near $e^{i\theta}$ on the ray from 0 to $e^{i\theta}$, the pseudohyperbolic disk $D(a,r)$ is contained in $\Gamma_\alpha(e^{i\theta})$. The disk $D(a,r)$ is a euclidean disk whose euclidean radius is comparable to $1-a$. For a close enough to $e^{i\theta}$,

$$z \in D(a,r) \text{ implies } |z - e^{i\theta}| < \delta.$$

Hence $\sup_{z\in D(a,r)}|g(z)| < \varepsilon\|g\|$. This violates (iii), so ν must be trivial, and $I_g \equiv 1$.

A similar argument handles the outer function O_g. If for all $\varepsilon > 0$ there exists e^{it} such that $|O_g^*(e^{it})| < \varepsilon$, we apply Lemma 4.5. The upper Carleson square in Lemma 4.5 contains some pseudohyperbolic disk that violates (iii), so O_g^* is essentially bounded away from 0. There exists $\eta > 0$, such that $|O_g^*(e^{it})| \geq \eta$ almost everywhere. Note $1/O_g \in H^\infty$, since for all $z \in D$,

$$\log|O_g(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|O_g^*(e^{it})| \frac{1-|z|^2}{|e^{it}-z|^2}\,dt \geq \log\eta.$$

We have reduced the symbol to a function $g = BF$, where $F, 1/F \in H^\infty$ and B is a Blaschke product, say with zero sequence $\{w_n\}$. We will show that the measure $\mu_B = \sum(1-|w_n|^2)\delta_{w_n}$ is a Carleson measure, implying B is a finite product of interpolating Blaschke products. (see, e.g., [13, Lemma 21]) Let $r < 1$ and $\eta > 0$ be as in (iii), so $\sup_{z\in D(a,r)}|B(z)| > \eta$ for all a. Given any arc $I \subseteq \partial D$, we may choose a_I and z_I such that $D(a_I, r) \subseteq S(I)$, $z_I \in D(a_I, r)$, $|B(z_I)| > \eta$, and $(1-|z_I|) \sim |I|$ as I varies. $\mu_B(S(I)) = \sum(1-|w_{n_k}|^2)$ where the subsequence $\{w_{n_k}\} = \{w_n\} \cap S(I)$. Assume without loss of generality that $|I| < 1/2$, so $|w_{n_k}| > 1/2$ for all k. This ensures $|1-\overline{w}_{n_k} z_I| \sim |I|$. Thus we have

$$\begin{aligned}
\frac{1}{|I|}\sum_k (1-|w_{n_k}|^2) &\sim \sum_k \frac{(1-|z_I|^2)(1-|w_{n_k}|^2)}{|1-\overline{w}_{n_k} z_I|^2} \\
&= \sum_k 1-(\rho(z_I, w_{n_k}))^2 \\
&< 2\sum_n 1-\rho(z_I, w_n) \\
&\leq -\sum_n \log \rho(z_I, w_n) \\
&= -\log \prod_n \frac{|w_n - z_I|}{|1-\overline{w}_n z_I|} \\
&= -\log |B(z)| \leq -\log \eta.
\end{aligned}$$

This shows μ_B is a Carleson measure.

(i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi)

Bourdon shows in [5, Theorem 2.3, Corollary 2.5] that (i) is equivalent to the reverse Carleson condition in Corollary 4.4 above, hence (i) $\Leftrightarrow$ (iv). This reverse Carleson condition also characterizes boundedness below of M_g on weighted Bergman spaces by Proposition 4.3. Thus (iv) $\Leftrightarrow$ (v). (v) $\Leftrightarrow$ (vi) is evident from the differentiation isomorphism (2.3). $\square$

4.2.2 Concluding Remarks

We suspect the results about H^2 can be extended to all H^p, $1 \leq p < \infty$, but without the Littlewood-Paley identity the proof is more difficult. Generalizing the results on Bloch to the α-Bloch spaces can be done with adjusted test functions as in [19]. Finally, we have partial results concerning S_g being bounded below on $BMOA$, but have not completed proving a characterization like the one in Theorem 4.7.

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