Controlled Navigation

Grade Levels

Intended for 2nd - 5th grade students, but may be expanded to higher grades.

Objectives

To understand the principles of mathematical proofs and reasoning while analyzing control theoretic-type problems in the form of mazes.

Materials and Resources

- Deck of Cards (optional)
- Mazes found in the worksheets PDF (Built in Google Docs)

Introduction

The goal of this activity is for the students to navigate a set of mazes from the upper left corner to the lower right corner. The only allowed moves are to travel horizontally or vertically from one grid point to another.

The introductory problem is to navigate a 5×5 grid with no restrictions. It should be emphasized that to move from one point to another costs 1 (initially), and that they should calculate a running tally up to arrival at the endpoint. Mention at this time the similarity between this problem and walking/driving through an area like downtown (or their typical walk/ride home). Navigating through these mazes in an optimal (shortest/least weighted path) manner can be applied to drive through downtown of a large city while trying to minimize the distance travelled, gasoline used, or time.

Encouraged conversations are:

1. Is there some strategy for choosing the path?
2. Is there a path that leads from the start to finish? (does a solution exist?)
3. Is it the only path? (is the solution unique?)
4. If not, is one of the paths the best? (is there an optimal solution?)

Problems

Students are provided the mazes and must work through them answering the questions above for each maze. They should also clearly mark their final path(s) and the cost to travel that path(s). Students should work individually as many small groups of them will produce unique/optimal paths and can thus be compared/contrasted against their peers.

Discussion

There are a variety of discussions that can occur:

1. What are strategies to solve a given maze? This can include arguments about traveling (or not traveling) certain directions on the map, arguments of traveling a given route versus another based on values, working backward rather than forward because working forward may be overtly complicated, solving sub-grids (2 $\times$ 2, 3 $\times$ 3, and 4 $\times$ 4 grids) to solve the main problem, etc.

2. Starting from unrestricted 2 $\times$ 2 grid, determine the number of possible paths (without backtracking) from the starting point to the endpoint. Work through the larger grids such as the 3 $\times$ 3 and 4 $\times$ 4. Is there a pattern? Can you use the process of induction to make predictions on larger grids?

3. What is a guaranteed way of finding the optimal path (if it exists) for the unrestricted direction problem? Here, since they have found all possible paths and corresponding costs, you can order them to find the optimal one(s).

4. Suppose we take two 5 $\times$ 5 grids and connect them endpoint of one grid to the starting point of the other grid. What can we say about the optimal path of this combined graph? Does this answer change if we fill in the remaining grid space to create a full 10 $\times$ 10 grid?

5. If we have the optimal path for a 5 $\times$ 5 grid and then reduce the grid to a 4 $\times$ 4, what can we say about the optimal path now?

If your students are very bright, there are plenty of other possible discussions. For example:

1. Working with larger grids. There is no limitation to a grid size, and the analysis started here leads to working with more difficult problems.

2. Introduction of the idea of metrics. Here, we measured distance not by calculating typical Euclidean distance between two points via the distance formula $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, but rather by traveling along horizontal/vertical movements of unit length, implying that the distance of any path is strictly a positive integer.

3. Each path length can be changed to accommodate the level of the students. For example, students learning addition/subtraction for their first time can be presented with mazes using whole numbers, for 3rd-5th grade students, the paths can be labeled with fractional or mixed fractional values, and for algebra students and above, the paths can be labeled with linear functions (such as $(x-5)$ or $(x+3)$ for example) and they can analyze the path values based on the choice of x.

4. Analyzing grids with arbitrary beginning and endpoints. All of the questions asked before are valid for this type of problem, but the problem is open for exploration for the different types of grid problems. Even working in a single 5 $\times$ 5 grid and varying the beginning/endpoints can be quite a difficult problem.

5. Require solving the same problems but with a waypoint(s), making a given grid point(s) required to be part of the solution path. All of the previous questions are still valid for this new problem.

Wrap Up or Alternative to Handouts

For those who don't want to work strictly with paper handouts, you can use a deck of cards to build random $n \times m$ grids (only limited to the number of cards you possess). In this scheme, you deal out the size of the grid you want to use. You can choose for the directions to be unrestricted except for blockades created by certain cards. Here, the value to travel from one card to the next would be the absolute value of the difference of card values. Here, the ace takes the value 1, all other numbered cards are their face value, and face cards (jack, queen, and king) are blockades. This way, you can create random grids at will, and you have complete control over beginning/endpoints.