Euclid's *Elements* Workbook

August 7, 2013

Introduction

This is a discovery based activity in which students use compass and straightedge constructions to connect geometry and algebra. At the same time they are discovering and proving very powerful theorems. The activity is based on Euclid's book *Elements* and any reference like "p1.4" refers to Book 1 Proposition 4 in *Elements*.

Section 1 introduces vocabulary that is used throughout the activity. Section 2 consists of step by step instructions for all of the compass and straightedge constructions the students will need. Section 3 contains all of the statements that we assume are true, most of which the students should be familiar with. Section 4 contains the statements that we wish to decipher and prove are true using our construction techniques. Section 5 is where the students begin their journey. Note the references in the parenthesis to the right of each activity, these are hints for the students.

1 Definitions

(d1) When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right*, and the straight line standing on the other is called a *perpendicular* to that on which it stands.

(d2) A figure enclosed by three straight lines is called *trilateral*.

(d3) A figure enclosed by four straight lines is called *quadrilateral*. In particular, a *square* is right-angled and equilateral and a *rectangle* is right-angled but not necessarily equilateral.

(d4) *Parallel lines* are indefinite straight lines which do not meet each other in either direction.

(d5) A *parallelogram* is a quadrilateral whose two sets of opposite sides are parallel.

2 Constructions

(c1) Copy a Straight Line:

- Adjust the compass to the size of the given line.
- Mark a point off the line.
- Place the compass on the new point and draw an arc.
- Draw a straight line from the point to any point on the arc.

(c2) Copy an Angle:

- Mark a point that will be the new angle's vertex.
- Draw a straight line in any direction from the new vertex.
- Place the compass on the vertex of the given angle and adjust it to any length.
- Draw an arc that intersects both lines which make up the given angle.
- Place the compass at the new vertex and draw the same arc intersecting the new line.
- Place the compass on the point where the first arc intersected one of the lines on the given angle and adjust the compass to the point of intersection on the other line.
- Without changing the width of the compass, place it on the point where the second arc intersected the new line and draw an arc that intersects the second arc.
- Draw a straight line from the new vertex to the point where the second and third arcs intersect.

(c3) Perpendicular Bisector:

- Place the compass at one end of the given line.
- Adjust the compass to slightly longer than half the line length.
- Draw arcs above and below the line.
- Keeping the same compass width, draw arcs from the other end of the line.
- Draw a straight line between the point where the arcs intersect above the line and the point where the arcs intersect below the line.

(c4) Extreme and Mean Ratio (Golden Ratio):

- Construct a square with the given line (see activity 3).
- Bisect one side of the square and mark this point.
- Draw a straight line connecting the bisector point and one of the opposite vertices.
- Place the compass at the bisector point and adjust it to the length of the previous drawn line.

- Draw a semi circle in the direction of the bisected line.
- Extend the bisected line to the arc.
- Adjust the compass to the length of the extension of the bisected line.
- Without changing the width of the compass, place it at one end of the given line and draw an arc intersecting the line.

3 Postulates

(p1.4) If two trilaterals have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, and so the trilaterals will be equal.

(p1.15) If two straight lines cut one another, they make the vertical angles equal to one another.

(p1.27) If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another.

(p1.29) A straight line falling on parallel straight lines makes the alternate & interior angles equal to one another.

(p1.34) In a parallelogram the opposite sides and angles are equal to one another, and the diameter bisects the area.

4 Propositions

(p2.1) If there be two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments.

(p2.4) If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.

(p1.35) Parallelograms which are on the same base and in the same parallels are equal to one another.

(p1.41) If a parallelogram has the same base with a trilateral and be in the same parallels, the parallelogram is double of the trilateral.

(p1.47) In right-angled trilaterals the square on the side opposite the right angle is equal to the squares on the sides containing the right angle.

(p2.11) To cut a given straight line so that the oblong contained by the whole and one of the segments is equal to the square on the remaining segment.

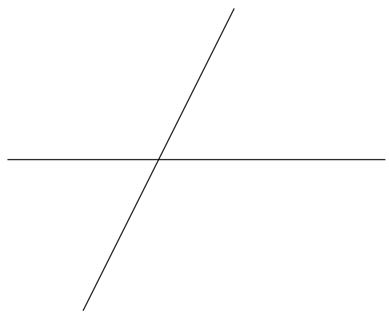
5 Activities

1. Construct a line perpendicular to the given line. (c3)

2. Construct a trilateral with the three given lines. (c1)

3. Construct a square with the given line. (c1,c3)

4. Construct a straight line parallel to one of the given lines. (c2, p1.27)



5. Construct your interpretation of Book II. Proposition 1. Argue why the proposition is true. Explain the proposition algebraically by labeling the different segments and describing the proposition with an equation. (c1,c3)

6. Construct your interpretation of Book II. Proposition 4. Argue why the proposition is true. Explain the proposition algebraically by labeling the different segments and describing the proposition with an equation. (c1,c3)

7. Construct your interpretation of Book I. Proposition 35. Argue why the proposition is true. (c1,c2,p1.4,p1.27,p1.34)

8. Construct your interpretation of Book I. Proposition 41. Argue why the proposition is true. Compare the formula for the area of a trilateral and the formula for the area of a parallelogram and relate it to this proposition. (c1,c2,p1.27,p1.34,p1.35)

9. Construct your interpretation of Book I. Proposition 47. Explain the proposition algebraically by labeling the different segments and describing the proposition with an equation. *Can you prove that this is true with what we have already done?*(c1,c3,p1.4,p1.41)

10. *Can you argue why it is possible to cut a line as in Book 2. Proposition 11. using the pythagorean theorem?* This is known as the extreme and mean ratio, or more commonly the **golden ratio**.(p1.47)

Notes for the teacher

- To see a proof of the pythagorean theorem as presented in *Elements* go to http://aleph0.clarku.edu/ djoyce/java/elements/bookI/propI47.html.
- To see a proof of cutting a line in extreme and mean ratio go to http://aleph0.clarku.edu/ djoyce/java/elements/bookII/propII11.html.
- For more information about the extreme and mean ratio (golden ratio) see http://www.mathsisfun.com/numbers/golden-ratio.html.
- Being able to cut a line in extreme and mean ratio leads to many other constructions, in particular constructing a regular pentagon. To see a construction of a regular pentagon using the extreme and mean ratio go to https://www.youtube.com/watch?v=bBPM4FvtnNU.
- For more motivation and excitement see http://www.georgehart.com/slide-togethers/slide-togethers.html to see what you can do with your regular pentagons. Note that the side lengths are also in extreme and mean ratio with the slits. Construction is not straightforward but see https://www.youtube.com/watch?v=dO6MznOU5DU for reference.