Gauss' Addition Method

Grade Levels

This activity is intended for grades 6 and higher.

Objectives and Topics

Students will learn how to find the sum of consecutive, positive integers between any two positive integers. By calculating the sums by hand, students can be lead to recognize a pattern using multiplication facts and develop their own general rules.

Introduction

Over the past few centuries, a lot of stories have arisen about famous mathematicians. One of our favorites is the story about the early life of Karl Friedrich Gauss. The version of the story we know goes like this: One day when Gauss was just a hoy, probably no older than age ten, his teacher started to get a migraine. To get some quite time, the instructor told the class to add up all the numbers from one to one hundred, figuring this would take at least an hour. Ten minutes later, young Gauss informed the instructor that he was done. Not only that, his answer was correct, the sum of all the whole numbers from one to one hundred is 5050.

Like all great mathematicians, Gauss was not only tremendously gifted, but also fundamentally lazy. Rather than do the rote work assigned, he took the time to find some clever way out of doing any hard work. He noticed that $100 + 1 = 101$, $99 + 2 = 101$, $98 + 3 = 101$, and so on. Finally, one gets $50 + 51 = 101$. Gauss then realized that adding all the whole numbers from one to one hundred givens the same sum as adding fifty 101s together, which is also the same amount given by $50 \cdot 101 = 5050$.

This shortcut to getting the sum of all the numbers from one to one hundred generalizes to any sum of consecutive whole numbers. Try adding up all the numbers between 1 and 23 by copying Gauss' method before reading on.

The natural place to start is by checking that $23 + 1 = 24$, $22 + 2 = 24$, and so on. Now we just need to know how many 24s we are able to make like this. We probably ran into a glitch when it was realized that 12 doesn't get "paired off" with anything. There are a couple of ways around this. We could just compute $24 \cdot 11$ and then add 12 or we could use Gauss' method to add all the numbers between one and 22, and then add 23. If we use this latter method, we see that the sum of all the numbers from one to 23 is $23 \cdot 11 + 23 = 23 \cdot 12 = 276$. Still, the former method works just as well, $24 \cdot 11 + 12 = 276$.

There is no reason we can't use this method when starting from a number other than one either. Try using Gauss' method to find the sum of all the numbers from 13 to 39 before reading on.

Notice that $13+39=52$, $14+38=52$, $15+37=52$, and so on. If you keep going, you find that the sum is equal to the sum of 13 copies of 52 and a left over 26. Using the former of the two rules above, we may calculate the answer: $13\cdot 52+26=702$. Here is a statement of a general rule:

> Let x and y be positive whole numbers. Find the difference between x and y, where $x>y$. The sum of all the whole numbers from x to y is equal to:
>
> $(x+y)\cdot(\frac{x-y}{2})+\frac{(x+y)}{2}$ if $x-y$ is an even number,
>
> $(x+y)\cdot\frac{x-y+1}{2}$ if $x-y$ is an odd number.

If we don't understand why these formulas always work, we can do some more examples and it should start to become clear. This is the most important part of your preparation.

Materials and Resources

Worksheet included in this PDF.

Teachers' Notes and Discussion

Some difficulties will occur if the students do not have some experience with multi-digit multiplication. However, if you feel it is appropriate for our classroom, you should have the use of calculators by students can be of aid.

This lesson offers an excellent opportunity for students to practice their skills at adding and multiplying numbers with many digits. The fact that their work is heading toward a creative end will help motivate this practice. It also allows students to participate in developing an advanced procedure for adding easily and computing complicated sums, helping to illustrate the relationship between the operations of addition and multiplication.

For this activity, the class should be split into small groups, six to a group at the very most, though groups of four or five work better. Each group will need something to record their work (a sheet of butcher paper for example). We can use whatever visual aids we like to demonstrate calculations (overhead, chalkboard, etc.). You will also need a large numbers of smaller objects and a container to put them in, such as marbles and a jar, tiles and a box, or even sheets of paper in a drawer. Use whatever is handy.

Start at a convenient spot in the classroom and have a student come put one object in the container. Choose another student and have them put two objects in the container. The third students should then put three objects in the container, etc. Keep going until every student in the classroom has placed objects in the container. It is easier to do the first problem with an even number of summands, so if there are an odd number of students, we can place one last set of objects in the container ourselves after all the students have gone.

Ask the class if they know a way to find out how many objects are in the container. It may take some discussion, but eventually the students should come up with the idea to sum 1,2,3, and so on up to the number of students in the class. Have the students work in small groups to find the sum. Tour the room to keep an eye on their progress and to make sure each group remains on task. Make sure they are recording their process. If the class commonly uses other resources with math problems, grant access to these as well.

This is the part of the exercise that lets students practice their strategies for addition. The students may come up with some innovative strategies for adding so many summands together. Rearranging some of the numbers to produce multiples of ten is a common approach. Some groups may simply try to use "brute force" and do the addition as $1+2=3$, $3+3=6$, $6+4=10$, and so on. This is okay. Some groups might even figure out Gauss' method themselves; this is also great! Above all, prepare for the students to take some unusual approaches and be ready for questions you think may arise.

After most of the groups have either finished or are close to doing so, have each group present their answer and how they got to it. Chances are at least some of the groups tried rearranging the summands to make "nice" numbers to add up. Write out the sum so that they entire class can see it (on the board, on an overhead project, etc.), and ask the students to work in their groups to try to rearrange the numbers to make them easier to add. Don't give the students quite as much time for this as the last task, but go around and monitor their progress. When reaching the point where a discussion can be had as a whole class, have the students stop and let each group explain how they tried to rearrange the numbers. Have a class discussion bout what is good or bad about each method and whether the students think there is an easier way. It is quite surprising how nice some of the students' ideas really are!

If any of the groups actually came up with Gauss' method, then after all groups have presented and class discussion has reached a stopping point, try to lead the class discussion into turning the addition problem into one of multiplication. If none of the groups came up with the method on their own, point out how the first and last numbers add to the same amount as the second and second to last numbers and so on; ask the students to work in groups to use this pattern to figure out the sum. Then, ask the class if the problem can be turned into one of multiplication and discuss how this will get the correct answer.

At this point, have each group tackle another sum using the method you demonstrated. Then, have the class come together and check each answer. Now, the students are ready to work on their own. If you have time, you can assign the worksheet as class work and monitor the students' progress, or assign it for homework.

Either way, review the worksheet as a group the next time the class meets. Discuss as a class whether or not the methods shown in the lesson and the worksheet will work for any sum, and for what kinds. The last portion in particular may be very challenging. If any of the students came up with the method given earlier for adding an odd number of summands, have them demonstrate and discuss as a class how to use the method for any consecutive sum of an odd number of summands. If none of the students found the solution presented here, have them break into groups again. Demonstrate that the problem can be done by summing all the numbers from 1 to 22 and adding 23 to that sum. After each group works out the solution using this method, bring the class back together and discuss how to add an odd number of consecutive summands.

After the class has arrived at Gauss' solutions for summing consecutive integers, be sure to summarize what they learned in this lesson. Turning it into a multiplication problem can more easily solve the problem of adding of consecutive numbers. The way that the multiplication works is different if there are an even or odd number of summands; it doesn't matter which number is started at, as long as the numbers are consecutive. Perhaps, students may put this method in their notes for future reference.

Gauss' Addition Method Worksheet **Name:** ____________________________

1. Find the sum of: $22 + 23 + 24 + 25 + 26 + 27 + 28 + 29 + 30 + 31 + 32 + 33 + 34 + 35$. Show all your work and explain why you solved this problem with the method you chose.

2. (a) What is $22 + 35$? How about $23 + 34$ and $24 + 33$?
   (b) Find the sum from problem 1 above using the multiplication method. Explain why your answer is correct.

3. (a) Find the sum of the numbers from 1 to 33. Show your work and explain how you solved this problem.
   (b) How was this problem different from the other problems you did in class and problem 1? Did you do anything differently to answer this problem? If so, explain.