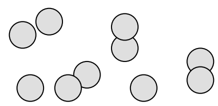
SUPER-M Fibonacci Nim

Introduction

The game of *Nim* usually refers to a one-on-one game where players take turns removing counters from community pile(s) of counters. By doing an Internet search for "Nim" it is immediately clear that there are numerous versions of the game. The version presented here is commonly referred to as "Fibonacci Nim" due to the curious connection between the optimal strategy for the game and the Fibonacci numbers. This game is intended for K through 4th grade.



Game Objective & Rules

Start with one pile of 10 counters. The objective of the game is to be the player who removes the last counter. (Equivalently, if there are no counters left and it is your turn, you lose.) Players alternate removing counters from the pile, according to the following rules:

- On your turn, you must take at least one counter.
- On the first turn, Player 1 can not take all the counters.
- On all following turns, you can take up to twice as many as the previous player took.

Here is an example of how a game might play out:

Player	Counters Removed	Counters Left	Comments
-	-	10	Game starts with 10 counters
1	3	7	First player can't remove all 10
2	1	6	Player 2 can remove up to 6
1	2	4	Player 1 can remove up to 4
2	4	0	Player 2 can remove up to 4
			Player 2 wins!

Find an Optimal Strategy

- Have pairs of students play the game of the Nim verse each other.
- Tell them there is a strategy which **guarantees** that they win, and challenge them to find it.
- Ask them to make observations about ways to win.

- See which students use paper and pencil to start figuring the game out. If none do, suggest or require it.
- Mathematicians like trying to solve simpler problems and using the solution to solve harder ones. If they are stuck, ask them if they can figure anything out if they start with fewer counters. Does that help them solve the game with 10 counters?
- Eventually, students should notice that if there are **three** counters left on their turn, they're going to lose (assuming they can't take all three). With more time, they'll notice the same thing about five **five** counters, and perhaps even **eight**.
- Challenge the advanced students by having them start with more than 10 counters. Can they figure out the pattern? If 3, 5, and 8, are bad numbers to be stuck with, what do they suppose the next highest "bad number" is?

Solution

The answer, of course, is the Fibonacci sequence. If a player can remove enough counters so that the number of remaining counters is a Fibonacci number, that player should be able to win. Note, the player has to get to the Fibonacci number in a "smart" way, meaning they don't take so many that their opponent can take all the counters.

So, when starting with 10 counters, Player 1 can **ALWAYS** win, and their best first play is to remove two counters, so that there are eight counters left, a Fibonacci number.

The opponent could then remove one, two, three, or four counters. If the opponent removes three or four counters, Player 1 can take the rest and win. If the opponent removes one or two counters, Player 1 can remove two or one counter(s) (respectively), so that there are five counters remaining (again, a Fibonacci number).

The opponent would not be able to remove all of the remaining counters, and if they removed any more than one counter Player 1 could take the rest and win. If the opponent only took one counter, there would be four left, meaning Player 1 could remove one counter also, so that there are three counters remaining (again, a Fibonacci number).

Still, the opponent could not remove all the counters. Now, regardless of if the opponent removes one or two counters, Player 1 could remove the rest, and therefore win!

Fibonacci Nim with More Counters

It turns out the same strategy as with 10 counters can be applied to games starting with any number of counters, with a slight modification in some cases. For reference, here are nine of the lower Fibonacci numbers:

$$3, 5, 8, 13, 21, 34, 55, 89, 144, ...$$

To illustrate the slight modification from a 10-counter game, consider a game starting with 20 counters. The next lowest Fibonacci number would be 13, so Player 1 may think they should remove seven counters to get the pile of 20 down to 13. But if Player 1 removes seven counters on their first turn, then their opponent could remove the rest! The point is that although we still want to get to Fibonacci numbers, we have to be careful how we get there. Luckily, it really isn't any more difficult than what we've done so far. Consider this:

Player 1 would like to make it so there are 13 counters left, which means seven counters need to be removed. Obviously, Player 1 does not want to remove seven on their first turn. Instead, Player 1 should consider removing the seven counters a mini-Nim-game within the actual-Nim-game. That is, how could Player 1 win a game with seven counters? Well, we've already seen that the best move would be to take two counters, so that their would be five counters left (a Fibonacci number). Regardless of the opponent's next moves, Player 1 would then be in the position to get the pile to 13 without removing so many that the opponent could win. So, using what we could refer to as a "7-game," Player 1 was able to reduce 20 counters to the Fibonacci number 13.

But then what? If the opponent removes two or more counters on their next turn, Player 1 could safely remove enough counters to get the pile to the next highest Fibonacci number eight. But if the opponent only removes one on their next turn, the pile would have 12 counters, and removing four would cost Player 1 the game. Similarly to what we did to get from 20 to 13, we can do to get from 12 to 8 play a "4-game." That is, remove one counter so that there are three (a Fibonacci number) remaining. Regardless of the opponent's next moves, Player 1 would then be in the position to get the pile to eight without removing so many that the opponent could win. Then from what we did above, Player 1 knows how to win the game.

Example 20-Game:

Player	Counters Removed	Counters Left
-	-	20
1	?	

Player 1 wants to get to 13, so they play a 7-game:

Player	Counters Removed	Counters Left
-	-	7
1	2	5
2	1	4
1	1	3
2	2	1
1	1	0

So, the 20-game actually looks like this:

Player	Counters Removed	Counters Left
-	-	20
1	2	18
2	1	17
1	1	16
2	2	14
1	1	13

Player 1 has successfully gotten the pile to 13.

Player	Counters Removed	Counters Left
2	1	12
1	?	

Player 1 wants to get to 8, so plays a 4-game:

Player	Counters Removed	Counters Left
-	-	4
1	1	3
2	1	2
1	2	0

So, the 20-game actually looks like this:

Player	Counters Removed	Counters Left
-	-	12
1	1	11
2	1	10
1	2	8

Player 1 has successfully gotten the pile to 8, and knows how to finish the game from here. Player 1 wins!

This idea of embedding smaller games inside large games allows a smart player to solve a game with Fibonacci number. In that case, clearly Player 2 has the advantage.

Expectations

The real point of this activity is for students to practice their critical thinking. While playing, students' conversations should be rich with "if-then" statements and logical arguments. Also, requiring students to do some writing will force them to formalize their reasoning. Recognizing that the optimal strategy has to do with the Fibonacci numbers is not as important as the basic problem solving skills required to get to that point. In fact, students may not know what the Fibonacci numbers are, but even recognizing that the odd numbers are the important numbers, have them test seven. Perhaps this will lead to the discovery that eight is the next important number. Some students may even guess that the number after eight is 13, without even knowing they've stumbled upon the Fibonacci sequence!

Common Core Standards

- Kindergarten:
  - K.CC.2 – Count forward beginning from a given number within the known sequence (instead of having to begin at 1).
  - K.CC.4 – Understand the relationship between numbers and quantities; connect counting to cardinality.
  - K.OA.1 – Represent addition and subtraction with objects, fingers, mental images, drawings, sounds, (e.g., claps), acting out situations, verbal explanations, expressions, or equations.

- 1st Grade:
  - 1.OA.1 – Use addition and subtraction within 20 to solve solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.
  - 1.OA.5 – Relating counting to addition and subtraction (e.g. by counting on 2 to add 2).
  - 1.OA.6 – Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8+6=8+2+4=10+4=14$); decomposing a number leading to a ten (e.g., $13-4=13-3-1=10-1=9$); using the relationship between addition and subtraction (e.g., knowing that $8+4=12$, one knows $12-8=4$; and creating equivalent but easier or known sums (e.g., adding $6+7$ by creating the known equivalent $6+6+1=12+1=13$).
  - 1.NBT.1 – Count to 120, starting at any number less than 120. In this range, read and write numbers and represent a number of objects with a written numeral.
- 2nd Grade:
  - 2.OA.2 – Fluently add and subtract within 20 using mental strategies. By end of Grade 2, know from memory all sums of two one-digit numbers.
  - 2.OA.3 – Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.
  - 2.NBT.5 – Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.
- 3rd Grade:
  - 3.OA.1 – Interpret products of whole numbers.
  - 3.OA.3 – Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.
  - 3.OA.9 – Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations.
- 4th Grade:
  - 4.OA.5 – Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself.
  - 4.NBT.4 – Fluently add and subtract multi-digit whole numbers using the standard algorithm.