Heron's Formula

Grade Levels

This lesson is meant for high school Geometry students, shortly after they have learned the Pythagorean Theorem, and the standard area formula for a triangle. The lesson also involves the reduction of square roots and can be useful for early Algebra students reviewing prime factorization.

Objectives and Topics

Students should learn that the Pythagorean theorem only works for right triangles, and that the standard base and height formula for the area of a triangle only works when you can find the vertical height. This lesson touches on the less often emphasized *contrapositive* to the Pythagorean theorem; Namely the fact that if $a^2 + b^2 \neq c^2$ then the triangle was not a right triangle to begin with. Students will use this to prove that certain triangles are scalene, thus motivating the need for a new area formula. The famous Heron's Formula comes to the rescue as a method which finds the area of *any* triangle, scalene or otherwise.

Materials and Resources

All that is required for this lesson is writing utensils and a large enough Black (or White) board so that the entire class can see.

There is a short list of so-called "Heronian triples" which will allow you to quickly give new examples that have nice whole number solutions. If side lengths are chosen at random the formula will still work, but the resulting area will most likely have square roots in its simplest form.

If the focus is not on Algebra and time is limited, then you may want to let the students use calculators to compute the square roots.

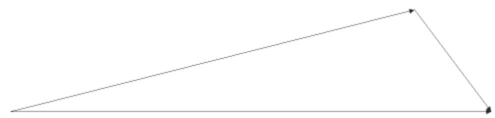
Introduction and Outline

Lead in to the discussion by talking about how to find the area of a triangle using the formula

$$\frac{1}{2} \cdot \text{(base)} \cdot \text{(height)}$$

Review with them that this formula only works when the height is perpendicular to the base. Draw a triangle on the board with side lengths 4, 13, and 15. The figure on the following page shows what the triangle looks like.

It is not necessary to draw this perfectly to scale, but it should be fairly obvious that it is *not* a right triangle. Ask the students what the area of this triangle is. If they respond with the above formula, ask them what



height they should be using. Remind them that they can only use 4 as the height of the triangle if that side is perpendicular to the base.

At this point some students will say that it isn't perpendicular, while some might argue that it is. To those who say it is not perpendicular ask,

"How do you know that it's not?" and if it comes up,

"Can you prove it without a protractor?"

Attempt to lead this line of questioning back to the Pythagorean theorem if possible. Explain that the Pythagorean theorem works for right triangles *only*. Ask the students to try out $a^2 + b^2 = c^2$ for this triangle. They should come to

$$4^2 + 13^2 = 16 + 169 = 185 \neq 225 = 15^2$$

It should be clear at this point that this is in fact *not* a right triangle, otherwise the theorem would work. Talk briefly with them about how the logic works in this situation. When the hypothesis and conclusion of the theorem are negated (not true), then the implication goes in the reverse direction. This can be shown in the diagram

$$\text{"right triangle"} \quad \Longrightarrow \quad a^2 + b^2 = c^2$$

$$\text{"}\textit{not}\text{ a right triangle"} \quad \Longleftarrow \quad a^2 + b^2 \neq c^2$$

Once this point has been made, congratulate them on proving that this is in fact a scalene triangle. Ask them if they would like to learn a formula that works for all triangles, right or otherwise. Ask,

"Wouldn't it be nice if there was a formula that always told you the area, no matter what the side lengths were?"

This is where you come to the rescue with Heron's Formula.

$$A = \sqrt{s \cdot (s-a) \cdot (s-b) \cdot (s-c)}$$

The s is Heron's formula is what's called the "semiperimeter". This is simply the perimeter, divided by two. Because this might seem strange to some of the students, relate the word to other "semi-" words like semicircle, semifinal, etc.

Explain that the formula works best if done as a process:

step $\#1$	$P = a + b + c$
step $\#2$	$s = \frac{P}{2}$
step $\#3$	find $s - a$
step $\#4$	find $s - b$
step $\#5$	find $s - c$
step $\#6$	multiply the results from steps 2 - 5
step $\#7$	take the square root

When done in this way, the formula is actually quite efficient. For the triangle in question we get that

$$P = 4 + 13 + 15 = 32$$
$$s = \frac{32}{2} = 16$$
$$s - a = 16 - 4 = 12$$
$$s - b = 16 - 13 = 3$$
$$s - c = 16 - 15 = 1$$
$$\sqrt{16 \cdot 12 \cdot 3 \cdot 1} = \sqrt{16 \cdot 4 \cdot 3 \cdot 3 \cdot 1} = \sqrt{8^2 \cdot 3^2} = \sqrt{8^2} \cdot \sqrt{3^2} = 24$$

This is where prime decomposition comes in handy. If you wish to focus more on the geometry than the algebra, you can have them use calculators to find the square root instead of reducing the radical. If the Algebra is your aim, then this is where you should review divisibility rules, and the fact that square roots distribute over multiplication (as in the last step).

Because there are many steps in the formula, it is best to do one more example before turning them loose on some problems. Suggest a $3, 4, 5$ triangle. The process is

$$P = 3 + 4 + 5 = 12$$
$$s = \frac{12}{2} = 6$$
$$s - a = 6 - 3 = 3$$
$$s - b = 6 - 4 = 2$$
$$s - c = 6 - 5 = 1$$
$$\sqrt{6 \cdot 3 \cdot 2 \cdot 1} = \sqrt{6^2} = 6$$

Now ask them if this is a right triangle. Some should recognize it as such, but you can now ask them to prove it. Remind them that they can use the other formula if the Pythagorean theorem holds true. Use the

Pythagorean theorem to prove that it is a right triangle, then ask them to use the regular base and height formula to check that the answer is correct. These steps should be done by students on the board if you can convince them to do so.

for your reference, the solutions are

$$3^2 + 4^2 = 9 + 16 = 25 = 5^2$$
$$6 = \frac{1}{2} \cdot 3 \cdot 4$$

So it works!

Problems

Below is a short list of Heronian triples that can be used for further examples or for homework

Side a	Side b	Side c	$s =$ semiperimeter	Area
5	5	6	8	12
5	5	8	9	12
5	12	13	15	30
9	10	17	18	36
7	15	20	21	42

The first three can be verified using the standard base and height formula, and the last two cannot.