Geometric Folds: Taking it to the Limit

Grade Levels

This activity is intended for junior or senior level high school students.

Objectives and Topics

In this activity, students will be introduced to the notions of a geometric sequence and calculate its limit by folding a band (or strip of paper) in a specific manner.

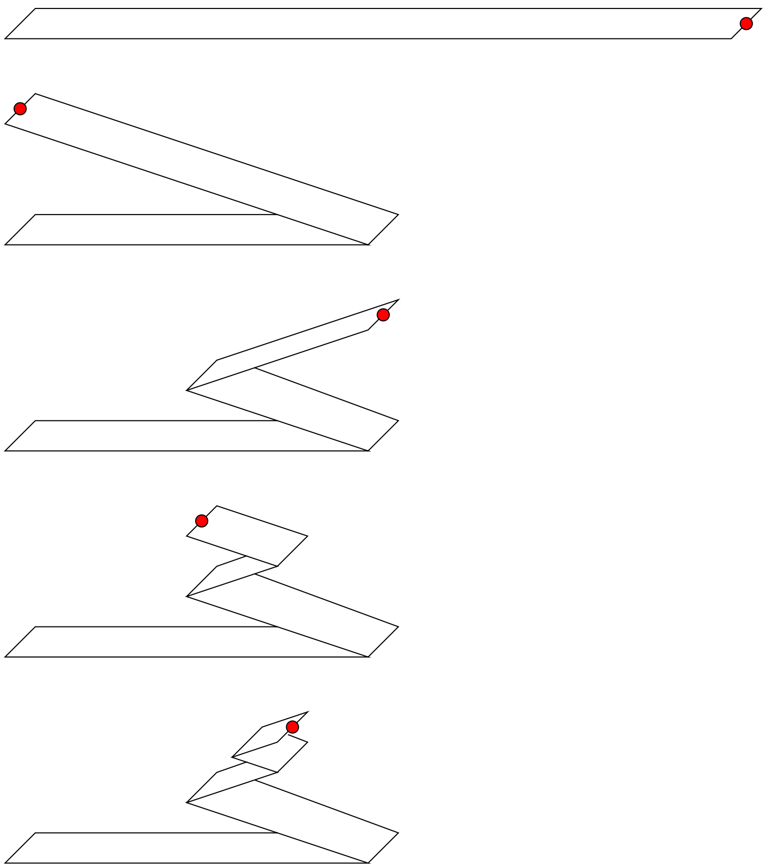
Materials and Resources

- Band or strip of paper marked by millimeters (longer the better)

Outline

Have the students begin working either alone or in small groups of 2 or 3. The students begin by folding their band according to the illustration. Then, they calculate each position of the point. As the band is folded, always alternating to the left and right, one notes that it is an alternating sum of powers of 12. The purpose is to see the elements of the synthesis of the possible approaches. At the end of the activity, do a collaboration so that the students can compare and explain their procedures and solutions.

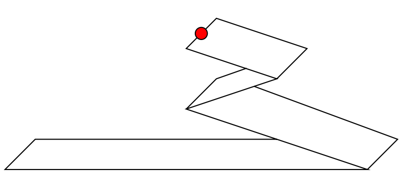
Folding



Etc... Where will the dot end up?

Element of the Synthesis

We suppose that the band is given to be length 1. One method for finding the dot's location is to express the sum of the different parts of the band, for example:



$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{8} = \frac{1}{4}$$

One is thus able to write the position of the point after n iterations (with$n > 1$ and where $n = 0$ is the unfolded band), which is given by:

$$S_n = \sum_{i=1}^{n-1} (-1)^{i-1} \frac{1}{2^i} = \sum_{i=1}^{n-1} (-1)^{i+1} \frac{1}{2^i} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots + (-1)^n \frac{1}{2^{n-1}}$$

Now, let's find the sum of S_n. For this, there are multiple approaches that lead to the same calculation:

1. Adding $S_n + \frac{1}{2}S_n$

$$\begin{array}{rcll} S_n & = & \frac{1}{2} & -\frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} + \cdots + (-1)^n \frac{1}{2^{n-1}} \\ +\frac{1}{2}S_n & = & & \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots + (-1)^{n-1} \frac{1}{2^{n-1}} \quad + (-1)^n \frac{1}{2^n} \\ \hline \frac{3}{2}S_n & = & \frac{1}{2} & \qquad\qquad\qquad\qquad\qquad\qquad\qquad + (-1)^n \frac{1}{2^n} \end{array}$$

where

$$\frac{3}{2}S_n = \frac{1}{2} + \left(-\frac{1}{2}\right)^n = \frac{1}{2} + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)^{n-1} = \frac{1}{2}\left(1 - \left(-\frac{1}{2}\right)^{n-1}\right)$$

Finally

$$S_n = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^{n-1}\right)$$

2. Factoring out $\frac{1}{2}$, one discovers a geometric sequence of the ratio $r = -\frac{1}{2}$:

$$S_n = \sum_{i=1}^{n-1} (-1)^{i-1} \frac{1}{2^i} = \frac{1}{2}\sum_{i=1}^{n-1} (-1)^{i-1} \frac{1}{2^{i-1}} = \frac{1}{2}\sum_{i=0}^{n-2} \left(-\frac{1}{2}\right)^i$$

Now, we have a sum of the form:

$$2 \cdot S_n = \sum_{i=0}^{n-2} (r)^i$$

where $r = -\frac{1}{2}$.

One can also use the same method as above to calculate $2S_n$, or suggest directly multiplying $2S_n$ by $1 - r$. In the latter case,

$$(1 - r)\sum_{i=0}^{n-2} (r)^i = 1 - r^{n-1}$$

finally creating

$$2S_n = \sum_{i=0}^{n-2} (r)^i = \frac{1 - r^{n-1}}{1 - r}.$$

Plugging back in $r = -\frac{1}{2}$, we get the same result as above

$$S_n = \frac{1}{2}\sum_{i=0}^{n-2} \left(-\frac{1}{2}\right)^i = \frac{1}{2}\frac{1 - \left(-\frac{1}{2}\right)^{n-1}}{1 - \left(-\frac{1}{2}\right)} = \frac{1 - \left(-\frac{1}{2}\right)^{n-1}}{3}$$

We remark that this approach permits us to directly generalize the problem studied to geometric sequences of any ratio r. Note that as n goes to infinity, S_n tends to $\frac{1}{3}$. Applied to our problem, as we fold the strip, the red dot will tend towards $\frac{1}{3}$ of the total original length of the strip.