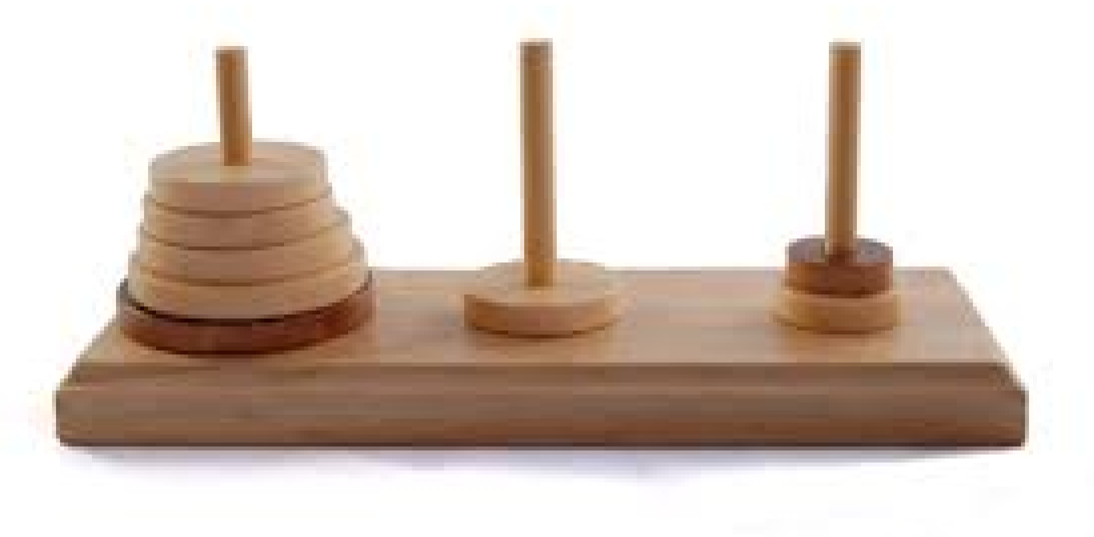
# The Towers of Hanoi

## Introduction

The Towers of Hanoi is a puzzle that has been studied by mathematicians and computer scientists alike for many years. It was popularized by the western mathematician Edouard Lucas in 1883.



The puzzle originates with a legend. This legend comes in various forms, so you may encounter a slightly different version if you were to research the Towers of Hanoi puzzle. The legend states that there is a secret room in a temple containing three large posts. The first post contains 64 golden disks stacked upon it, starting with the largest disk on the bottom, and decreasing in size to the smallest disk on top of the stack. Monks (or priests) have been been moving the disks continuously since the beginning of time. The goal is to recreate the 64 disk tower on the third post. The monks must move the disks according to two rules:

1. The monks can only move one disk at a time.
2. The monks can only place smaller disks on top of larger disks.

Once the monks make the final move and complete the puzzle, the world will end.

## Grade Level

This activity and student handout were designed for students in grade 9-12. However, this activity could be adapted for lower grade levels as well. The puzzle itself is suitable for all grade levels. Elementary school students can develop logical reasoning skills by playing with the puzzle, and middle school students can begin to develop strategies to solve the puzzle optimally.

## Objectives and Topics

The mathematics concepts utilized in this activity are pattern recognition, sequences, recursion, algorithms, binary number system, number system conversions, place-value, functions and algebraic manipulations, unit and measurement conversion, exponents, exponential growth and geometric sum formulas.

Students will work in groups to develop strategies and algorithms for solving the puzzle. They will determine both a recursive and and explicit formula for finding the minimum number of moves required to solve an $n$-disk puzzle. They will solve the legendary puzzle, and compute how long it will take for the “world to end".

## Materials

Students will need the Student Handout Packet, a pencil, and a calculator for the final computation. They will also need some form of the Towers of Hanoi puzzle:

- This activity was implemented with a puzzle created out of cardboard boxes. 6 cube-shaped boxes in decreasing size (filled with some sand for stability) were taped shut and wrapped with wrapping paper. (2 alternating colors work best, as this visual helps the students develop accurate strategies.) Three X's were taped to the floor, indicating the location of the “posts".
- Amazon sells many Towers of Hanoi sets
- There are many free iPad and Android Towers of Hanoi applications. A quick Google search will also produce many computer applications, for example:

  `http://lawrencehallofscience.org/java/tower/tower.html`

- Anything stackable in decreasing size, such as a pile of coins, will also work.

## Activity

The entire activity could run through a few class periods. However, not every activity is necessary, and the teacher can choose to skip or shorten some parts as necessary. Refer to the Student Handout and Student Handout Solutions as well.



### Introduction

5 minutes
Introduce the legend, the puzzle, and the rules.

### Hanoi Races

20-30 minutes
Have students work in groups to complete the 6-disk puzzle.

### Optimal Strategies

30-45 minutes
In order to determine how long it will take for the monks to finish the puzzle, an optimal strategy is needed. Introduce and discuss the idea of an algorithm, and optimality. The teacher can give examples of other, more familiar algorithms (i.e. tying shoes, cooking a recipe, PEMDAS).

Students will work on finding an optimal algorithm for the $n = 4$ disk puzzle, and write it out as completely as possible. Once students are finished, have groups switch algorithms. Students must then use ONLY the other group's algorithm to solve the puzzle.

**Discussion Questions:** How was the other group's algorithm the same as your group's? How was it different? Was anything about the other group's algorithm unclear or confusing? Did it lead you to the optimal solution?

### Binary Number System

25-30 minutes
This section* is not absolutely necessary for solving the puzzle, but it is a nice introduction into algorithms, and the results can give great insight on solving the puzzle. It also introduces the Binary Number System and number conversions. It may be skipped, however, if there is lack of time.

Introduce Binary Number System. Practice converting back and forth between Base-10 and Base-2 (more practice problems than the ones provided in the Student Handout can easily be created for further practice.)

**Discussion Questions:** What are the pros and cons of a Base-10 Number System? Of a Base-2 Number System? Why do we use Base-10? Why do computers use Base-2? What properties of the Towers of Hanoi puzzle are binary?

### Binary Number System Strategy

30-40 minutes
Introduce the algorithm that uses the Binary Number System to solve the puzzle (written out completely in the Student Workbook. It may take students a few tries to get the hang of it. Encourage students to go slowly step-by-step, and think about which part of the algorithm they need to follow at each step. Have the students use the algorithm to complete the puzzle for $n = 2, 3,$ and $4$ disks (or more if time permits!)

**Discussion Questions:** Do you notice any patterns? When does the algorithm stop? Does this match your optimal solutions that you found before? What's similar about the algorithm for the $n = 3$ disk puzzle and $n = 4$ disk puzzle? Can we start to generalize the minimum number of moves for an $n$ disk puzzle using this strategy?

### Recursion and the End of the World

30-60 minutes
Introduce sequences and function notation. Introduce recursion and recursive sequences. Utilize the Fibonacci Sequence as an example. (If students are familiar with the factorial operation, then the teacher can also use that as an example of a recursive function.) The teacher can also draw and use tree-diagrams to help describe recursion. Talk about the two main components of a recursive function.

Students will work on developing a recursive function $f(n)$ to represent the minimum number of moves required to solve the $n$-disk puzzle. Have them verify their formula for $n = 1, 2, 3,$ and $4$ against their previous work, and practice drawing tree-diagrams.

Students will then work towards developing an explicit function $f(n)$. Encourage them to start with $f(n)$, and work backwards by replacing $f(n-1)$ with it's recursive definition, and so-forth. This is certainly not an easy task, and the students might need some assistance, especially towards the end. Students should start to see powers of 2 appearing, and eventually start to see a sequence that looks like $f(n) = 2^{n-1}+2^{n-2}+\cdots+2^3+2^2+2^1+2^0$. At this point, introduce the idea of a geometric sum, and the formula used to evaluate sums of this form.

Finally, have students evaluate $f(n)$ for $n = 64$, to find the minimum number of moves, and convert the moves into some type of time measurement (i.e. 1 move = 1 second) to determine how long it will take for the world to end. Concluding remarks can be made about exponential growth and large powers of 2.