Optimal Cone

1 Grade Levels and Time

- Grades:11-12
- Time: This lesson will take two 50-minute class periods.

2 Objectives and Topics

- Objectives:
  - The students should be able to formulate the volume of the cone based off of the angle α (See Graphic under Procedure).
  - Students should be able to use ruler and protractor to collect data and calculate the area of the cones they make.
- Topics:
  - Data Collection
  - Polynomial Regression using TI, Desmos, or other graphing utilities.
  - Pythagorean theorem
  - Arc Length of a circle
  - Calculating derivatives using product and chain rule
  - Finding the max of a function by calculating critical values.

3 Introduction

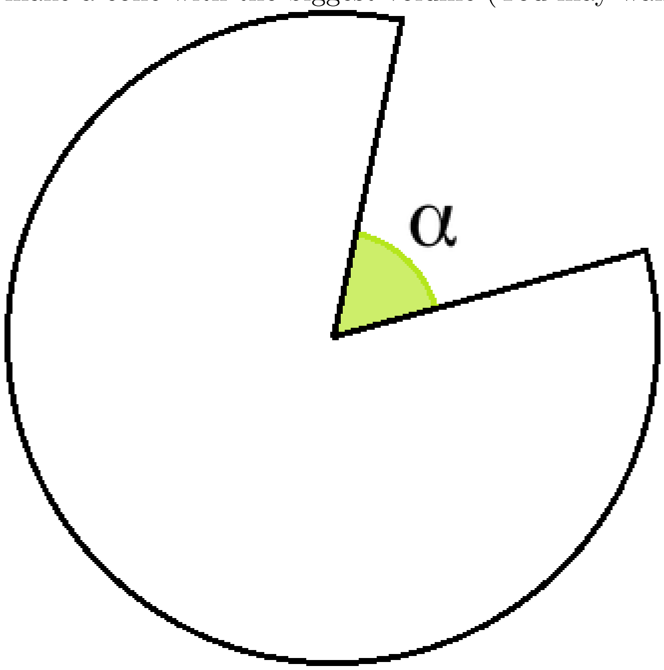
Collecting and plotting data can give students a guide as to what the correct answer might be. In this activity, students will create a cone from a paper disk and determine how to maximizes the volume by analysing numerical data and then by using calculus. This lesson also gives the students a hands on way to formulate a real problem.

4 Procedure and Discussion

1. EXPLORATION
   (a) Begin by handing out Circular cut outs to the students, located in the Materials section. Either have the students cut out the circle and cut the dotted line or pre-cut the circle and cut the dotted line.

(b) Have the students practice making a cone with the cut out.

(c) Now the students should have their cut out, ruler, protractor, and formula for calculating the volume of a cone. With the materials, have the students choose an angle *alpha* for which to make a cone with the biggest volume (You may want them to.



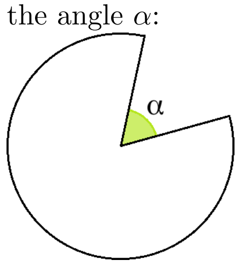
(d) Now collect the data and plot it on Desmos, which is a good site to use for this project. The first column should be the angle and the second column should be the cones volume.

(e) , then ask the students what type of equation (linear, quadratic, cubic, etc.) the data looks like. Then have them use the regression function on excel, TI, or other resource in order to find the curve of best fit.

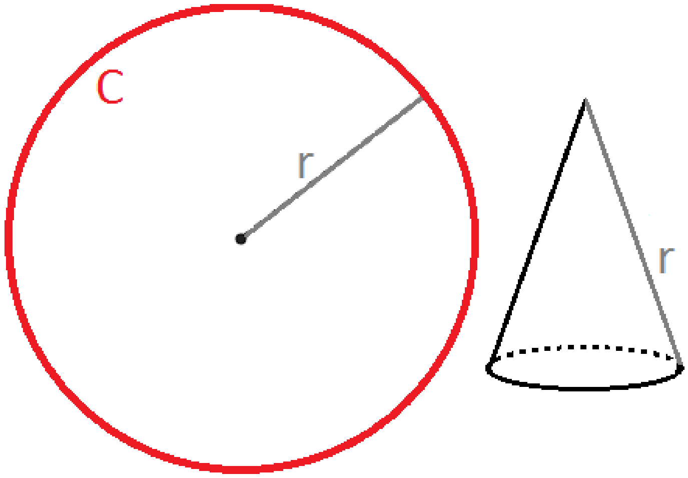
(f) Lastly with the portion, have the students use the curve of best fit to find the angle that gives them the largest volume and have the students discuss any questions they have and/or how they came up with their answer.

2. COMPUTATION

(a) Now the students should work on creating an equation for the volume of a cone with respect to the angle α:



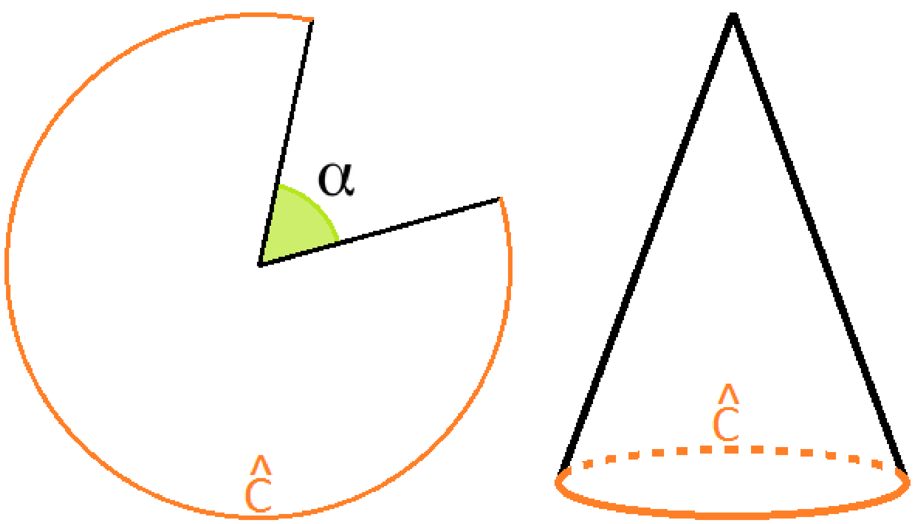
(b) Start by having them notice what is constant about their cut out disk. The slant height r of the cone is always constant and the starting circumference C of the disk is also constant.



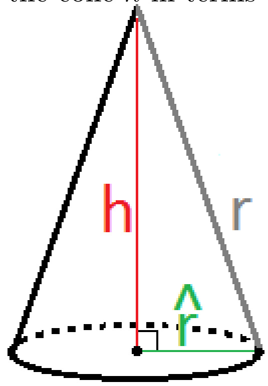
$$C = 2\pi r$$

(c) It should be easier for the students to figure it out the radius of the cone $\hat{r}$ once they find the circumference of the cone $\hat{C}$. Note:

$$\hat{C} = \frac{2\pi - \alpha}{2\pi} \cdot C = \frac{360 - \alpha}{360} \cdot C = 2\pi\hat{r}$$



(d) Now that the students know $\hat{r}$ then they can use the pythagorean theorem to put the height of the cone h in terms of $\hat{r}$.



$$h = \sqrt{r^2 - \hat{r}^2}$$

(e) From 2(c) we know that

$$\begin{aligned} \hat{r} &= \frac{2\pi - \alpha}{2\pi} \cdot \frac{C}{2\pi} \\ &= \frac{2\pi - \alpha}{2\pi} \cdot \frac{2\pi r}{2\pi} \\ &\boxed{= \frac{2\pi - \alpha}{2\pi} \cdot r} \end{aligned}$$

(f) Now we have enough information to plug everything into the volume formula for a cone.

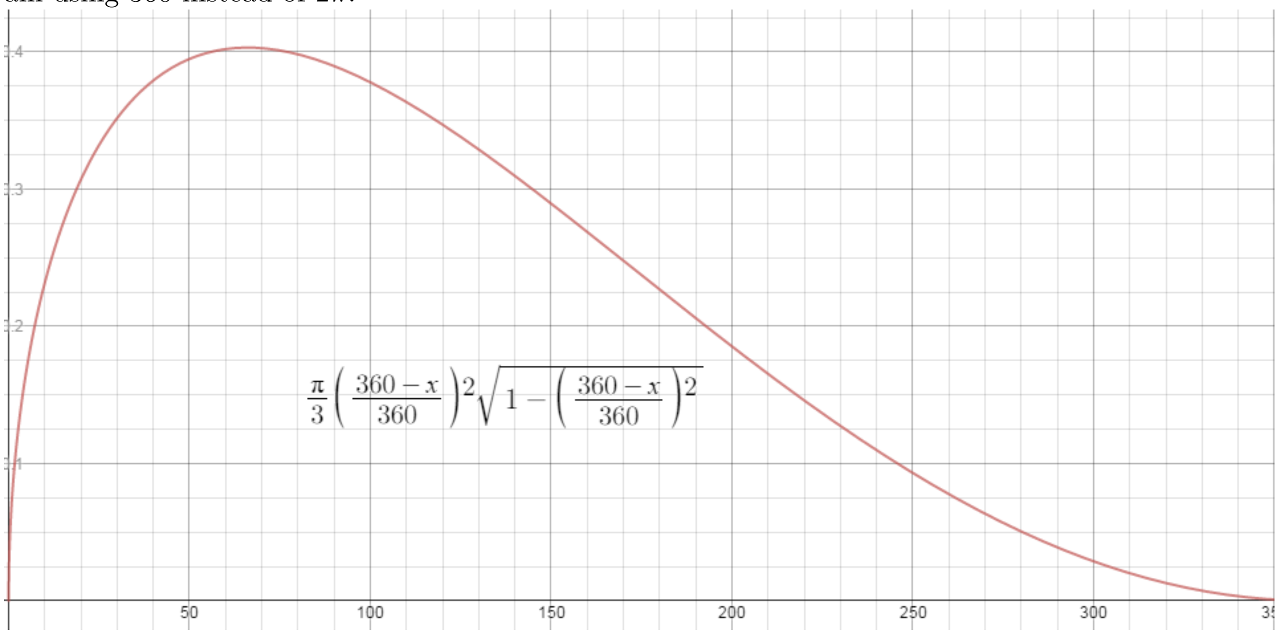
$$\begin{aligned} V &= \frac{\pi}{3} \cdot h \cdot \hat{r}^2 \\ &= \frac{\pi}{3} \cdot \left(\sqrt{r^2 - \hat{r}^2}\right) \cdot \left(\frac{2\pi - \alpha}{2\pi} \cdot r\right)^2 \\ &= \frac{r^2\pi}{3} \cdot \left(\frac{2\pi - \alpha}{2\pi}\right)^2 \left(\sqrt{r^2 - \hat{r}^2}\right) \\ &= \frac{r^2\pi}{3} \cdot \left(\frac{2\pi - \alpha}{2\pi}\right)^2 \left(\sqrt{r^2 - \left(\frac{2\pi - \alpha}{2\pi} \cdot r\right)^2}\right) \\ &= \frac{r^2\pi}{3} \cdot \left(\frac{2\pi - \alpha}{2\pi}\right)^2 \left(\sqrt{r^2\left(1 - \left(\frac{2\pi - \alpha}{2\pi}\right)^2\right)}\right) \\ &\boxed{= \frac{r^3\pi}{3} \cdot \left(\frac{2\pi - \alpha}{2\pi}\right)^2 \left(\sqrt{1 - \left(\frac{2\pi - \alpha}{2\pi}\right)^2}\right)} \end{aligned}$$

3. Now discuss with the students how we only care about α and if α maximizes the volume of a cone made from a cut out with radius of 1 unit then it maximizes a cone made from a disk of radius 1000

units. So, we can set $r = 1$ just to simplify things more.

$$= \frac{\pi}{3} \cdot \left(\frac{2\pi - \alpha}{2\pi}\right)^2 \left(\sqrt{1 - \left(\frac{2\pi - \alpha}{2\pi}\right)^2}\right)$$

4. Now the students have an equation that they can graph and compare to their data points. Desmos is a good site to use when graphing and plotting data to compare. Note that in the graphic below I am using 360 instead of 2π.



5. Finally, with Desmos the students can instantly find the critical point, which is at 66.1 degrees. Then students should find the first derivative and solve for the critical value that way. The optimal angle should be

$$-120(\sqrt{6} - 3)$$

-or-

$$-\frac{2\pi}{3}(\sqrt{6} - 3)$$

5 Materials