Mathematical Music Theory: Part I

Grade Levels

This activity is intended for grades $7-12$.

Objectives and Topics

The goal of this lesson is to apply the concept of modular arithmetic to transpose the key of a popular song. Math topics to be covered in this lesson include algebra and modular arithmetic.

Note: This lesson can be done in conjunction with the Modular Magic Pinwheel lesson also found on the SUPER-M Website. If done after the Modular Magic Pinwheel lesson, part of this lesson may be skipped or simply reviewed briefly.

Instructions

In this lesson, we will use arithmetic very similar to the arithmetic we use on a clock. Begin by answering the questions, disregarding A.M. and P.M. Think of the face of a clock. Again, we are disregarding A.M. and P.M., so we make no distinction between 12 A.M. and 12 P.M., just as we cannot discern this information by only looking at the face of a standard clock.

1. If it is $5:00$, what time will it be 5 hours later? What about 10 hours later?
2. How many hours can I add to the current time so that the time will remain the same? *Remember, we are considering A.M. and P.M. to be the same.*
3. Suppose it is $10:00$.
   (a) What time will it be in 14 hours? (The answer is $12:00$.)
   (b) Is there a number of hours from $0-11$ that you could add to $10:00$ that would yield the same time? (The answer is 2 hours.)
4. You should have observed from the previous problem that adding 2 hours is the same as adding 14 hours. What number from $0-11$ is the same as adding 16 hours? 20 hours? 25 hours?
5. Is a noticeable pattern emerging? Try dividing each number by 12 and listing the remainder. What do we notice?

Modular Addition

What you have just been doing is essentially a method called modular addition. In modular addition, we pick a number to treat like 0 and work from there. In the examples we have been doing, 12 is essentially 0 since whenever we add 12 hours, the time does not change (we are not considering A.M. or P.M.). We write this as $12 \equiv 0$ mod 12, and read "twelve is congruent to 0 mod 12." In this system, any multiple of 12 is the same as 0 mod 12. For example, $24 \equiv 0$ mod 12 and $60 \equiv 0$ mod 12. Similiarly, $13 \equiv 1$ mod 12 (since adding 13 is the same as adding 12 and 1, but 12 is the same as 0). Try the next few quick modular addition problems. Answers should always be in the form of a number from $0 - 11$ with "mod 12" at the end.

Examples

1. $5 + 9 = ?$ mod 12
   (a) The answer is 2 mod 12.
2. $7 + 10 = ?$ mod 12
   (a) The answer is 5 mod 12.
3. $4 + 8 = ?$ mod 12
   (a) The answer is 0 mod 12.

Mathematics and Music

In modern western music, there are 12 tones that are spaced equally (well equal is actually relative – it is based of a logarithmic scale). We call the scale made of all the notes the chromatic scale. The chromatic scale cycles back to itself so that the notes repeat again. The notes are as follows:

$$A \quad A\#/B\flat \quad B \quad C \quad C\#/D\flat \quad D \quad D\#/E\flat \quad E \quad F \quad F\#/G\flat \quad G \quad G\#/A\flat$$

Note that we began with A since it is the first letter of the alphabet, but sometimes musicians begin with C since the C major scale is the only scale to have no sharps or flats. Notice that there are two names for some notes. That is because, for example, $A\#$, read "A sharp", is the same note as $B\flat$, read "B flat." With each note of the scale we can associate a number. Write the following on the top of your paper:

A	$B\flat$	B	C	$C\#$	D	$E\flat$	E	F	$F\#$	G	$G\#$
0	1	2	3	4	5	6	7	8	9	10	11

Note that we picked a name for each of the sharp/flat notes according to standard notation, but you can choose them differently if you'd prefer.

Finally, we get to the desired goal! In music, we sometimes change the key of a song by keeping the relative distances between notes the same, but shifting everything either higher or lower. This is helpful when a song is beyond the vocal range of a singer, or when we want to convey a different attitude or feeling. Changing keys is also called transposing in music. We can think of chaining keys as simply adding a number. Try the following example:

Example

Let's say there is a chord progression of: $G \to C \to D$. This is represented by the number changes $10 \to 3 \to 5$. Shift the entire key up 2 by adding 2 mod 12.

Solution:

1. $10 + 2 = 0$ mod 12.
   (a) In our chart, 0 is associated with A.
2. $3 + 2 = 5$ mod 12.
   (a) In our chart, 5 is associated with D.
3. $5 + 2 = 7$ mod 12.
   (a) In our chart, 7 is associated with E.

The result of the transposition is $A \to D \to E$.

Try transposing the key of a popular song that you like by adding a number module 12. The following site has a plethora of popular song chords: www.ultimate-guitar.com