Mathematical Music Theory: Part II

Grade Levels

This activity is intended for grades $7-12$.

Note: This is the second part of the Mathematical Music Theory Lesson. Click here for part one.

Objectives and Topics

The goal of this lesson is to use modular addition to replace the concept of subtraction and apply this to lowering a key, followed by using modular addition to study scales. Math topics covered in this lesson include modular arithmetic and algebra.

Introduction

In the last lesson, we only considered transposing by adding. This is essentially equivalent to raising the key. What if we want to lower the key? What is, what if we wish to subtract? We will see that subtraction is the same as addition with the proper choice of numbers.

Clock Analogy

Let us return to the analogy of a clock. Again, forget about A.M. and P.M.; we are only concerned with the actual number of the hour. So 3 A.M. and 3 P.M. are the same for our purposes. Consider the following questions:

1. If it is $5:00$, what times was it 3 hours ago? Is there a number of hours we could add to get the same result?
2. If it is $3:00$, what time was it 3 hours ago? is there a number of hours we could add to 3 to get the same result?
3. If it is $12:00$, what time was it 3 hours ago? Is there a number of hours we could add to 12 to get the same result?
4. Do you notice a pattern? Is subtracting 3 always the equivalent to adding the same number?
5. Adding what number is the same as subtraction 1? What about 2? What about 3? What about 4?

In modular arithmetic, we do not even consider subtraction as an operation because it is superfluous. Subtraction can always be modeled by addition of the appropriate number. In particular, if we are working modulo 12, then subtracting a number n between 0 and 12 is the same as adding $(12-n)$.

6. Complete the following table:

Subtraction	Equivalent Addition mod 12
1	Answer: 11
2	Answer: 10
3	Answer: 9
4	Answer: 8
5	Answer: 7
6	Answer: 6
7	Answer: 5
8	Answer: 4
9	Answer: 3
10	Answer: 2
11	Answer: 11

Music

In music, it is often advantageous to lower a key. Let's practice by lowering the key of the following notes by "subtracting" by 5.

7. Lower the Key of $D \to A \to B \to G$ by 5.

First rewrite the following:

A	$B\flat$	B	C	$C\#$	D	$E\flat$	E	F	$F\#$	G	$G\#$
0	1	2	3	4	5	6	7	8	9	10	11

Solution:
We can begin by rewriting the above progression in terms of numbers. We have $5 \to 0 \to 2 \to 10$.
We wish to subtract 5, but we can think of this instead as adding 7. However, it may be advantageous to actually subtract in our heads instead of adding whenever the numbers are larger than 5. In either case, the result is the same. Note that the resulting key can either be thought of as lowered by 5 steps or raised by 7 steps. Both are exactly the same thing!

$5 + 7 \equiv 0$ mod 12
$0 + 7 \equiv 7$ mod 12
$2 + 7 \equiv 9$ mod 12
$10 + 7 \equiv 5$ mod 12

Thus, the resulting chords are $A \to E \to F\sharp \to D$

You can lower the key of a favorite song using the same method. Try lowering the key of a song you like by 7 steps.

Scales

We now shift our attention to scales, beginning with the most common scale, the major scale. The major scale is characterized by the intervals at which notes are located, beginning with the root note. The root note is simply the first note of the scale. For example, the A scale begins with A and includes notes at certain intervals from A. The notes in the A scale are $A, B, C\#, D, E, F\#, G\#$, and back to A. In terms of the way we are characterizing notes based on numbers, the notes in the A scale are $0, 2, 4, 5, 7, 9, 11$, and then to 12, which is the same as 0 mod 12. If we look carefully at the intervals, they are simply obtained by adding $2, 2, 1, 2, 2$, and 1, in that order, all mudulo 12. Every major scale is obtained in that manner.

Problem

1. What are the notes in the B scale?

Solution:
Begin by looking at the number for B, which is 2. We then add $2, 2, 1, 2, 2, 1$ modulo 12, to obtain the correct notes.

2
$2 + 2 \equiv$ **4** mod 12
$4 + 2 \equiv$ **6** mod 12
$6 + 1 \equiv$ **7** mod 12
$7 + 2 \equiv$ **9** mod 12
$9 + 2 \equiv$ **11** mod 12
$11 + 2 \equiv$ **1** mod 12
$21 + 1 \equiv$ **2** mod 12

The numbers for the B scale are $2, 4, 6, 7, 9, 11, 1$, and back to 2.

The associated letters are $B, C\sharp, E\flat, F, G, A$, and back to B.