Population Problems

Grade Levels

This activity was intended for grades $7 - 12$.

Objectives and Topics

The purpose of this lesson is to develop a model for population growth and to develop proficiency with algebraic equations. Math topics incorporated into this lesson are modeling, algebra, systems of equations with multiple variables, recursive equations and sequences.

Activity Length

This activity will take $1 - 2$ periods, depending on the depth. The notes on stage models may be useful as a lead-in to these problems.

The Questions

1. Suppose you start with a pair of bunnies (all pairs will be male/female mating pairs). If a pair of bunnies produces one pair of baby bunnies each month, and it takes baby bunnies one month to mature before they can begin to reproduce, how many bunnies will you have in a year?

2. Suppose now that a pair of bunnies produces *four* pairs of bunnies each month, and it takes baby bunnies *two* months to mature. How many bunnies will you have in a year?

To clarify

Just in case it's unclear how this works, I'll step through problem 1. I dart with a pair in January, let's call them pair A. They reproduce in January, producing a pair of babies (call them pair B). In February, pair A produces another pair of babies (pair C), while pair B does nothing because they're too young. In March, paid A AND pair B (now mature) produce a total of 2 pairs of babies, while pair C does nothing. Continue through December.

The Wrong Way

Most people will start by counting, taking a tally, and trying to methodically step-by-step track how many bunnies ail be produced in a year. This method can work – but it's easy to get confused and difficult to keep track of which bunnies are maturing and which bunnies are adults. Not to mention, how do you *neatly* track it all? Problem 1 is tricky enough to track these ways; problem 2 is a good way to drive your students insane.

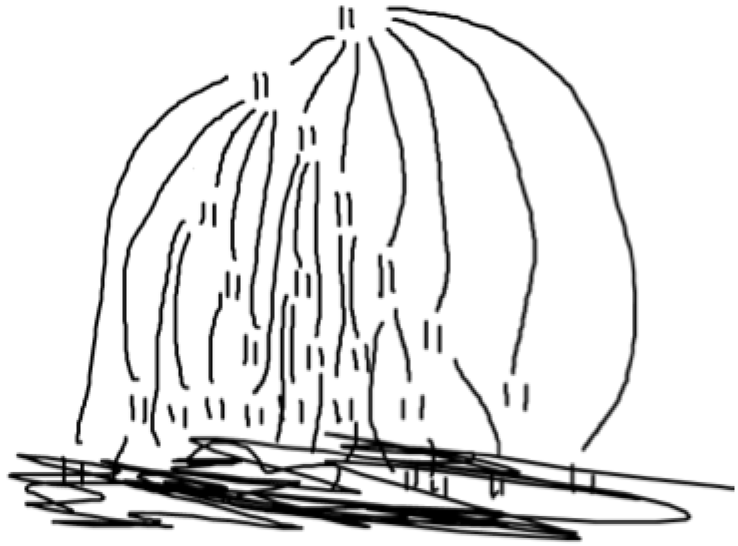

Figure 1: One common approach. Each row is a month with lines drawn to show parent-offspring connec- tion. Note that it gets messy (and frustrating) very quickly.

2	
2	2
4	2
6	4
8	6
~~10~~ ~~16~~ 14	~~6~~ 8

Figure 2: Another approach. Cleaner, with total adults and offspring being calculated in each column. If done carefully, this actually works. But notice that the approach is divorced from the actual biological situation - it's just numbers in a mystical pattern. So it's easy to get confused as to what you're supposed to add at each step. A single mistake cascades and can also lead to false patterns arising.

A Modeling Approach

First off, note that everything is in pairs, so unlike the previous examples, let's just count pairs as our unit (ie. we'll use "1 pair" instead of "2 bunnies"). That'll keep the numbers smaller. A stage model diagram might help here to capture exactly what's going on. Here's what the stage diagram will look like for problem 1.

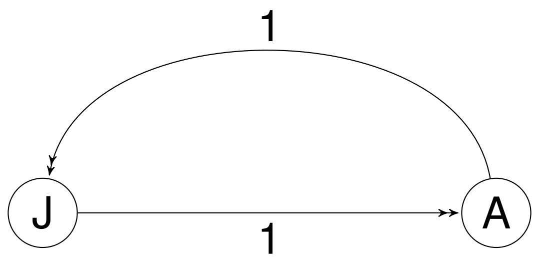

Figure 3: The stage diagram, showing Adults (A) and Juveniles (J). Juveniles mature into Adults every month. Adults produce one pair of Juveniles every month.

Once you have the diagram, the equations flow naturally. Each stages gets an equation, each arrow creates a term. So we get:

$$A(t+1) = A(t) + J(t)$$

"The number of Adults I'll have next month month is the number of Adults I currently have, plus the number of Juveniles who will mature in that time (which is all of the current Juveniles)." Notes that Adults giving birth have no effect on the Adult population, so that arrow actually has no term associated with it.

$$J(t+1) = J(t) - J(t) + A(t)$$

"The number of Juveniles I'll have next month is the number of Juveniles I currently have, minus the Juveniles who will mature in that time (which is all of the current Juveniles), plus the number of Juveniles who will be born in that time (which is equal to the number of current Adults)."

We can simplify a bit and get a pair of recursive equations:

$$A(t+1) = A(t) + J(t)$$
$$J(t+1) = A(t)$$

Using these equations, we can now calculate our populations. We start with $A(0) = 1$ (1 pair of Adult bunnies at the start) and $J(0) = 0$.

So in January:

$$A(1) = A(0) + J(0) = 1 + 0 = 1$$
$$J(1) = A(0) = 1$$

February:

$$A(2) = A(1) + J(1) = 1 + 1 = 2$$
$$J(2) = A(1) = 1$$

March:

$$A(3) = A(2) + J(2) = 2 + 1 = 3$$
$$J(3) = A(2) = 2$$

And we continue on with this pattern. The equations make is much easier to keep track of our population. If we continued, we'd find that the adult population has the following sequence: January:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...$$

Wait – that's the Fibonacci sequence! Careful thought of the problem should reveal why this is: "My addle population next month is equal to my current Adult population plus the number that mature, but the number that mature is equal to my Adult population from a month ago, because that's how many gave birth last month..."

Let's do it with math. First, a change of variable (think of it as rolling back the clock a month). The equation doesn't actually change: January:

$$J(t + 1) = A(t) \to J(t) = A(t - 1)$$

Now a substitution:

$$A(t + 1) = A(t) + J(t)$$
$$A(t + 1) = A(t) + A(t - 1)$$

That's the recursion equation for the Fibonacci sequence! This particular problem is said to have been proposed by Fibonacci and motivated the creation of this famous recursion relation.

So, the answer... In twelve months, I'll have 233 adult pairs. Juveniles can be calculated using our equations: $J(12) = A(11) = 144$. So I'll have 377 pairs of bunnies, or 754 bunnies total.

What about Problem 2?

Let's approach problem 2 in the same way. Start with a stage diagram. There are a couple ways to do it, but because of the two month maturation period, I'm going to split Juveniles into two parts – Newborns and 1-month old Juveniles.

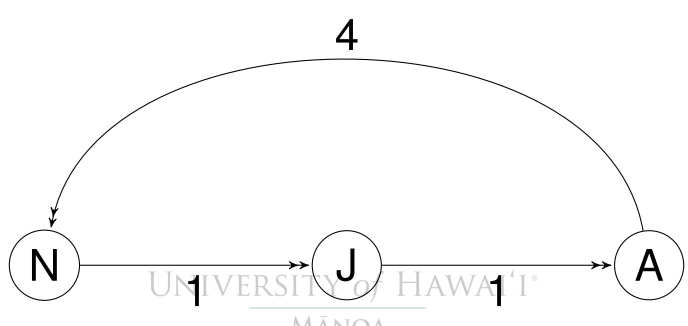

Figure 4: The stage diagram, showing Adults (A) and Juveniles (J). Juveniles mature into Adults every month. Adults produce one pair of Juveniles every month.

The equations flow just as before:

$$A(t+1) = A(t) + J(t)$$
$$J(t+1) = J(t) - J(t) + N(t) = N(t)$$
$$N(t+1) = N(t) - N(t) + 4A(t) = 4A(t)$$

You could solve it using these equations, or we could do some clever simplifications again. A change of variables:

$$J(t+1) = N(t) \longrightarrow J(t) = N(t-1)$$
$$N(t+1) = 4A(t) \longrightarrow N(t-1) = 4A(t-2)$$

And now some substitutions:

$$A(t+1) = A(t) + J(t)$$
$$A(t+1) = A(t) + N(t-1)$$
$$A(t+1) = A(t) + 4A(t-2)$$

So, it's not quite as simple as the Fibonacci sequence, but it's still not too bad. Simple arithmetic from here on out. I'll leave this one to you.

Extending It

Looking at the final recursion equations for Adult populations, you might conjecture how the recursion formula will look if I said Juveniles take 3 months to mature and Adults produce 6 pairs of offspring each month:

$$A(t+1) = A(t) + 6A(t-3)$$

And you'd be right. A general form would be $A(t+1) = A(t) + BA(t-M)$, where B is the number of offspring pairs born each month, and M is the maturation time.

For more of what a stage model can do, see the stage model notes posted on the SUPER-M website. If you do choose to make your model more complex, please note that simplifying the equations to a single recursion relation like we did may not be as possible. Only this particular situation allowed for easy manipulation of the equations. But you can still get solutions by using the full unsimplified system of equations.